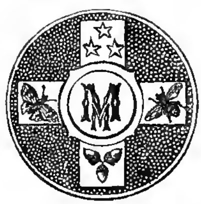
Prof. H. J. S. Smith
with sincere regards
from the author

TEMPERAMENT

R. H. M. BOSANQUET



AN ELEMENTARY TREATISE

ON

MUSICAL INTERVALS

AND

TEMPERAMENT

WITH AN

ACCOUNT OF AN ENHARMONIC HARMONIUM EXHIBITED IN THE LOAN COLLECTION OF SCIENTIFIC INSTRUMENTS SOUTH KENSINGTON, 1876

ALSO OF AN

ENHARMONIC ORGAN EXHIBITED TO THE MUSICAL ASSOCIATION OF LONDON, MAY, 1875

BY

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Freylich würde der Gesang noch mehr gewinnen, wenn wir die enharmonischen Töne in unserm System würklich hätten. Alsdenn würden sich die Sänger auch von Jugend auf angewohnen, die kleinsten enharmonischen Intervalle richtig zu singen, und das Ohr der Zuhörer, sie zu fassen; und dadurch würde in manchen Fällen der Ausdruck der Leidenschaften sehr viel stärker werden können.

Kirnberger, vol. i. p. 19.

Greater certainly would be the gain of Song if we really had the enharmonic intervals in our system. For then singers would accustom themselves, from their youth up, to sing correctly the smallest enharmonic intervals, and the ear of the listener to appreciate them; and thereby would it be possible, in many cases, to make the expression of the passions very much stronger.

PREFACE.

THE investigations of which this work contains an account have been published during the last two or three years in various forms*. But these forms were suitable for those who were already well acquainted with the subject, and the order of treatment was that which commended itself as correct in a scientific point of view; i.e. the matter being purely deductive, the general theory was stated first, and everything deduced from it. For more general purposes this arrangement does not appear to commend itself; and the general theory has now been relegated to such a position that the rest of the work may be read independently of it. The general treatment is, except in this Chapter (X) and the note to Chapter III, elementary throughout; and repetition is introduced as much as possible instead of reference; so that it may be hoped that the difficulty of the study is much reduced. The arrangement adopted is unsatisfactory from a scientific point of view. But the different parts of the subject are so intertwined, that if the correct order is once forsaken, it is impossible to separate them out into another satisfactory scheme.

The relations of technical music and musicians to this subject have until lately been, for the most part, of an almost hostile character. The facts are entirely unknown to musicians in general, and of the theory the wildest ideas have been formed. Of the objects which I place before myself, musicians generally form their own ideas, and stick to them. The 'Musical Standard,' after the reading of a

* Proc. Royal Society, June, 1875, p. 390; Philosophical Magazine, Jan. and Sept. 1875; Proc. of the Musical Association, 1874–5; Stainer and Barrett's Dictionary of Musical Terms, Art. 'Temperament.'

paper of mine at the Musical Association, solemnly announced that I proved on a huge black-board that the equal temperament scale was all wrong, when nothing of the kind was even said, much less proved on the black-board.

As emphatically as I disclaim all idea of proving the equal temperament scale wrong by 'black-board' considerations, so emphatically do I protest against the idea that it is my object to abolish the equal temperament. It is not the same thing to say, Let us make something new, to wit B; and, Let us abolish something old, to wit A. The two things, A and B, are not mutually exclusive; and there can be no reason why they should not flourish together. There is no force in the objection to the study of this subject founded on the idea that its tendency is towards the exclusion of the ordinary system. There will always be the proper applications of the one arrangement, and the proper applications of every other, just as there are the proper applications of the different instruments in an orchestra.

Dr. Stainer's* 'Harmony Founded on the Tempered Scale' is the only work on technical music, so far as I know, which takes up on this matter a position logically quite unassailable. The position may be put thus:—our music is made of certain artificial things, twelve notes dividing the octave equally; give us these, and let us examine their combinations and capabilities. This is a perfectly correct reduction of the harmony of the equal temperament to a science of classification; the various combinations are in fact enumerated and classified. Deeper than classification the work does not go.

The present work will be grounded on principles of a type precisely analogous. I shall not attempt to enter into the question of the physiological basis of harmony, or any of the

* Dr. Stainer informs me that this work does not represent his present opinions: it forms however an excellent text for discussion.

questions discussed by Helmholtz and others in connexion with this part of the subject. I avoid the controversy, not because I fail to have definite opinions on these points, but because they are quite distinct from what I am dealing with here.

I shall assume, as matter of experience, independently of any theory, that notes separated by certain musical intervals form smooth combinations when sounded together; and that the accurate adjustment of an interval admits of variation within certain narrow limits, without any serious injury to the effect of the smooth combination. The provision of material for such smooth or approximately smooth combinations, or harmonious chords, forms the object of these investigations. The law of vibration ratios is assumed, as being amply verified by experience, and on it is founded a reduction of intervals to equal temperament semitones, the theory of which I have endeavoured to explain in an appendix, in the hope of making it clear to those who are unacquainted with logarithms.

The proceeding of this treatise is therefore one of classification; but it is of a wider scope than Dr. Stainer's. That is a classification of the material of one system, the equal temperament. This is primarily a classification of systems, with remarks on a few points connected with the separate classifications, corresponding to Dr. Stainer's, of the material of some of the principal systems. The treatment appropriate to different systems differs widely. Such as are derived from perfect or approximately perfect fifths and thirds require different treatment from either the equal temperament, or the class of systems of which the mean-tone system is the type, and although this latter class admits of treatment by means of the ordinary notation, yet the practical results differ so far from those of the equal temperament that the best effects are produced by a different style of handling. For instance, in writing for the mean-tone system, it is advisable to avoid the employment of fifths very high in the scale, where their imperfection is most sensible.

The remarks made by musicians on enharmonic systems in general are for the most part characterised by misapprehension of the facts. For instance, at the beginning of Dr. Stainer's preface, it is stated that, if an enharmonic scale were feasible, doubtful chords could not exist, 'because mathematical correctness of ratio would make every chord strictly in tune in one key, instead of allowing it to be somewhat out of tune in several keys. The whole of our musical literature, from the works of Bach to those of Wagner, would therefore be unavailable for instruments with an enharmonic scale.' Now although it is true that the best applications of enharmonic scales must be those which are made with reference to the particular system employed, yet the particular objection made above is quite unfounded, at least from my point of view. A chord is in or out of tune quite independently of the key in which it is supposed to be. And, as a matter of fact, there is not the least difficulty in playing Bach on the mean-tone system. The difficulty that arises in adapting music to the class of systems which have perfect or approximately perfect fifths and thirds, arises from a different source*; it has nothing to do with the so-called enharmonic changes; the treatment of these rarely, if ever, presents difficulty; and it is generally practicable to arrange an enharmonic change so that suspended notes undergo no change in pitch. In fact the enharmonic change has reference to the relation with the preceding and following harmony, and does not generally affect the distribution of the chord itself. Enharmonic changes of pitch can however be occasionally used purposely with good effect; and the conjecture of Kirnberger in the motto on the opening page has been to some extent verified.

Again, in Dr. Stainer's preface we find, 'The tempered scale is certainly out of tune, and will not bear to have its proportions exhibited to an audience with better eyes than ears, on a white screen' [or black-board], 'but its sounds

* The false fifth in diatonic and allied scales; (d —\ a in the key of c.)

have nevertheless been a source of as real pleasure to all great composers, as of imaginary pain to certain theorists.' Now my own attention was first directed to this subject in an entirely practical manner, viz. by taking part in the tuning of my own organ (an ordinary instrument with two manuals and pedal). The process of trial and error employed in tuning the equal temperament on the organ throws into relief the effect of the equal temperament modification on the chords; and this, in the tuning at all events, is very disagreeable. I have never yet met any musician who was in the habit of personally taking part in or superintending the complete tuning of an organ, who did not agree that the imperfections of the present methods are startling when thus encountered, and that closer approximations to harmonious chords are a matter of great interest. I must say that musicians thus practically familiar with tuning are very rare.

Again: 'When musical mathematicians shall have agreed amongst themselves upon the exact number of divisions necessary in the octave; when mechanists shall have invented instruments upon which the new scale can be played; when practical musicians shall have framed a new system of notation which shall point out to the performer the ratio of the note he is to sound to its generator; when genius shall have used all this new material to the glory of art—then it will be time enough to found a Theory of Harmony on a mathematical basis.'

This admirable passage, which however contains some confusion of ideas, was of great use to me in directing attention to the principal points involved. The theory of the division of the octave has now been completely studied; a generalised keyboard has been invented and constructed upon which all the new systems can be played; a notation has been framed by which, in systems of perfect and approximately perfect fifths and thirds, the exact note required can be indicated, and it has been shown that other systems require no new notation. ('The ratio of the note to its generator' arises from the notion of a harmonic scale; but I have not used anything of the kind, and it is incompatible with derivation by division of the octave.) The new material may be therefore said to be ready. But the idea that the theory is to follow the practice is not true here; for in this case a somewhat extended view of the theory has been necessary to render the practice possible. This is the general course where a science has a practical side: the practical side is in advance up to a certain point in the history; the theory lags behind. But it may always be expected that at some point the theory may overtake the practice; and then, and not till then, is it capable of rendering useful assistance. Take the example of astronomy. In Newton's time Flamsteed, the observer, threw cold water on the theoretical treatment of the moon's motions; he said it never had been of any use, and never would be. But now where would our knowledge of this subject be but for the Lunar Theory? Theory, and theory only, has succeeded in so far converting the moon into a clock in the sky, that ships depend on this means primarily for ascertaining their longitude.

Throughout the foregoing I have employed the word *theory* in conformity with Dr. Stainer; but I do so under protest. Strictly speaking, Dr. Stainer's part of the subject, the harmony of any system, is not a theory at all so long as it is treated in the way in which he (quite correctly in my opinion) treats it. It is a classification. The word 'theory' includes the explanation of the facts by natural causes (the reference to mechanical and physiological laws); it would also, in its usual acceptation, include the deductive process here employed, the sense of the word in this application being analogous to its use in the expression 'Theory of Numbers.' But we cannot call the process of classification of combinations a theory, any more than we should call the classifications of botany a Theory of Flowers. It is, however, merely a question of the use of a word, and only becomes important when it conveys the idea that by the process of classification we have got to the bottom of the matter, a view which will be admitted to be erroneous.

On the question of practical application I may summarise shortly the principal points. I consider that the best application of the mean-tone system will be to the organ with a generalised keyboard of twenty-four keys per octave.

Of the positive systems, or those with approximately perfect fifths and thirds, which require a new notation, I have little doubt that the most obvious application, and one from which we are not far distant, is in the orchestra. Instruments are now being constructed with comma valves; in designing these, it must be always remembered that the comma deviations must not be from equal temperament, but from perfect fifths or perfect thirds. We have already a comma trumpet; some progress has been made with the clarionet, and on the whole it seems likely that we shall have the instruments ready before we have the compositions. The notation of this work is the only one hitherto proposed which is competent to deal with this question practically*. The example at the end shows how it may be used. With the violin it will be only a matter of study. Harmoniums, such as the large one now in the Exhibition at South Kensington, besides the interest and beauty of their effects, seem to be needed as means for the study of the combinations of these systems; for it will not be possible to use these intervals effectively in the orchestra unless the composer has made himself practically acquainted with their treatment.

The application of these perfect-fifth systems to the organ would increase the bulk of the instrument too much in proportion; and the instability of the tuning of organ pipes renders it very doubtful whether a proportionate advantage could be in this way obtained for a permanence. The mean-tone system is more suitable for the organ, as being less

* Mr. Ellis's notation of duodenes, although theoretically a solution, cannot be said to be practical, in my opinion.

sensitive in the tuning, requiring only double the ordinary number of notes per octave, and being remarkably easy of performance on the generalised keyboard.

As to the pianoforte, the only application that it is at all likely that it would be worth while to make is one of the mean-tone system with twenty-four keys to the octave. But it is only on the finest modern grand pianofortes that the equal temperament is really offensive; and as these instruments are generally used for purposes of display, when the quality of the chords is not heard at all, even this application cannot be regarded as likely to offer great advantages.

In discussing the origin of the mean-tone system, I have entered at some length into the history of the subject, especially as regards its relations with Handel and Bach. The statements about Bach conflict to some extent with received opinions, but they will be found to be well supported. The principal authority I have employed, which is not generally accessible, is a life by C. L. Hilgenfeldt, published by Hofmeister at Leipzig, for the centenary of Bach's death, July 28, 1850; a work of great completeness, frequently clearing up matters that are left obscure by Forkel as well as by later biographers.

In contemplating the imperfections great and small which the science of acoustics show us beset all our ordinary instruments, it is not unnatural to wonder that we get our music so tolerable as on the whole we do. An eminent musician, who objects strongly to acoustics in general, and to my investigations in particular, is in the habit of saying, that he never hears a lecture on acoustics without wondering that we have any music at all. Now what are the facts?

First who is to be the judge, and according to what standard?

As to who is to be the judge, I think that most musicians will agree that those who have a very high development of the sense of absolute pitch have their ears altogether more finely strung, and more acute, than other people. That is to say, if a man can tell me the exact sound of c and of any other note as he ordinarily uses them, without having any instrument to refer to, I consider that his musical organisation is such that his verdict on performances may be accepted without hesitation, so far as their being in or out of tune according to his standard is concerned.

As to the standard. The standard of all the musicians of this class with whom I am acquainted is the equal temperament; and I think that the limit of the distinct perception of error under ordinary circumstances is about $\frac{1}{10}$ of an equal temperament semitone. It is said that much smaller intervals can be distinguished, but I doubt whether this be the case under the unfavourable circumstances of public performances.

With my experience of first-hand accounts of performances from musicians of this character, it is quite as commonly the case as not that performances, even by artists, orchestras, or choirs of considerable reputation, are stigmatised as extremely defective in tune.

When we pass to the consideration of such errors as half a semitone, which any competent musician can detect, I say that it is rare to hear any performance of any kind in which errors of this magnitude are not occasionally committed; especially by ordinary string quartetts, ordinary choirs, and wind instruments in ordinary orchestras.

There are undoubtedly a certain number of fine organisations, whose instincts and great technical mastery guide them to a satisfactory result. And I take it, it is only in consequence of the facilities which exist for employing everywhere those who are thus eminent in their particular lines, that we have anything that can be called music in our public performances. Of course I am speaking only of instruments where the intonation is either made by the player, or depends on his care.

The method of which the present investigations form one branch, will try to meet the difficulty in question by making a special study of small variations in interval, with the view of adopting systematic methods for the attainment of what has hitherto only been possible with the assistance of great natural gifts.

The first thing to be attended to in this more general view of the subject is the effect of temperature.

It is not possible to go in detail into this subject here; I will only mention one thing. It admits of being proved that if the relation be assigned between the source of heat in action, and the means for its dissipation (radiation &c.), then there will be a temperature to which each portion of the space considered will rise, and at which it will remain steady; as much heat being parted with in every instant as is derived from the source. We may apply the principle to such an instrument as the clarionet. The steady temperature of this instrument must be but little below that of the breath. But whatever the steady temperature is under given circumstances, the rule should be, raise the instrument to the steady temperature before the performance begins, or rather before the tuning is effected. A cupboard at a regulated temperature would effect this completely.

The actual work done by me, which this treatise is intended to illustrate, consists of the construction of the harmonium and enharmonic organ, which are mentioned on the title page and described in their places. I have had frequent opportunities of letting visitors to the Loan Exhibition at South Kensington hear the harmonium, and the recognition of its success with respect to the purity of the chords is all that I could desire. The enharmonic organ was exhibited to the Musical Association immediately after it was finished, and has since that time stood in my rooms at Oxford where it now remains.

ST. JOHN'S COLLEGE, OXFORD,
1876.

CONTENTS.

CHAPTER I.

HARMONIOUS CHORDS.

CHAPTER II.

EQUAL TEMPERAMENT.

CHAPTER III.

PERFECT FIFTHS. PYTHAGOREAN SYSTEM.

CHAPTER IV.

THE GENERALISED KEYBOARD.

CHAPTER V.

MEAN-TONE SYSTEM. OLD UNEQUAL TEMPERAMENT.

CHAPTER VI.

APPLICATIONS OF THE MEAN-TONE SYSTEM.

CHAPTER VII.

HARMONIC SEVENTH.

CHAPTER VIII.

APPLICATIONS OF THE SYSTEM OF PERFECT FIFTHS AND ALLIED SYSTEMS.

CHAPTER IX.

ENHARMONIC HARMONIUM. ENHARMONIC ORGAN, POSITIVE STOP.

CHAPTER X.

GENERAL THEORY OF THE DIVISION OF THE OCTAVE.

CHAPTER XI.

MUSICAL EMPLOYMENT OF POSITIVE SYSTEMS HAVING PERFECT OR APPROXIMATELY PERFECT FIFTHS.

APPENDIX.

CHAPTER I.

HARMONIOUS CHORDS.

WHEN two musical notes are regarded relatively to each other, they are said to form an interval.

When two musical notes are sounded together, there are two principal cases which may occur:—

(1) The effect of the combination may be smooth and continuous;

(2) The combination may give rise to beats, or alternations of intensity or quality, more or less rapid.

(1) The principal intervals for which the effect is always smooth and continuous, whatever be the instrument employed, are known to musicians under names of the Octave, Fifth, and Major Third. These are called consonances. All other consonances can be derived from these. Fourths to the bass and Harmonic Sevenths may form also smooth combinations, but are not called consonances by musicians.

(2) If the interval between two notes be nearly but not exactly a consonance, or a harmonic seventh, beats may be heard.

The foundation of modern harmonious music may be said to be the smoothness of the consonant intervals, and of intervals which differ from these only by small magnitudes. The harmonic seventh has not yet been employed in modern music.

To provide a material of notes for musical performance it is theoretically requisite in the first place that to every note used we should possess octave, fifth, and major third, up and down. Each of these being a note used we may require the same accessories to each, and so on.

It is universally agreed that the octaves used should be exact; but for the most part the fifths and thirds and their derivatives are made to deviate from their exact values by small quantities. These deviations constitute 'Temperament.'

The interval of a fifth is easy to tune exactly with all ordinary qualities of tone; it is also easy to make a fifth sharp or flat by any given number of beats per minute. For this reason, as well as others, the relations of fifths are preferred for discussion.

Conception of a Major Scale.

The Major Scale may be conceived of as consisting of two sets of notes, first the key-note and notes immediately related to it by fifths, and secondly notes related to the first set by thirds. Thus in the key of C we have

F—C—G—D

related by fifths; and

A—E—B

which form major thirds above the first three.

Conception of a Minor Scale.

The Minor Scale may be conceived of as consisting of the four first notes, together with the major thirds below the three C—G—D; namely,

A♭ — E♭ — B♭

If we write down all these notes so that the fifths are counted upwards and major thirds sideways, we have the following scheme, omitting the two notes in brackets:—

B♭	D	[F♯]
E♭	G	B
A♭	C	E
[D♭]	F	A

If we then fill up the bracketed places, we have a scheme of twelve notes, corresponding in name to the twelve notes in ordinary use, but forming a system of consonant chords, including the principal chords of the keys of C major and minor, with some others. Such an assembly of notes Mr. Ellis calls a duodene*; and in particular the notes above written are said to constitute the duodene of C.

It will be seen that, by specifying the duodene in which any combination of notes is to be taken, the exact notes to be performed can be indicated; and the indications can be obeyed if the notes constituting all the duodenes required are provided. The system of duodenes forms a practicable method by which rigorously exact concords can be employed and controlled.

If however we provide all the notes necessary for an extended system of duodenes, we have endless series of fifths running up and down, and endless series of thirds running horizontally; and it is possible to show that no two of the notes will ever be exactly the same in pitch. Consequently in practice various approximations are employed, so as to reduce the number of notes required.

* Proc. Royal Society, Dec. 1874. The word may be taken to mean a set of twelve notes.

CHAPTER II.

EQUAL TEMPERAMENT.

THIS is the method of tuning in ordinary use for keyed instruments. The simplest way of considering it is to observe that the interval of any octave on keyed instruments is made up of twelve equal semitones, thus—

c – c♯ – d – d♯ – e – f – f♯ – g – g♯ – a – a♯ – b – c.

We shall regard equal temperament semitones simply as intervals, twelve of which make an octave. And we shall in future reckon intervals in equal temperament semitones (E. T. semitones). These are the semitones of the pianoforte and organ as ordinarily tuned. E. T. is used as an abbreviation for the words 'equal temperament.'

Perfect Fifths and Thirds.

It has been long known that octaves have the vibration ratio 2 : 1, perfect fifths the vibration ratio 3 : 2, perfect thirds the vibration ratio 5 : 4; also that the logarithms of vibration ratios are measures of the corresponding intervals. From these principles it is possible to show * that—

the perfect fifth is 7.01955 E. T. semitones (say $7\frac{1}{51}$);

the exact major third is 4—.13686 E. T. semitones (say $4-\frac{1}{7.3}$).

The E. T. fifth is seven semitones, so that the perfect fifth is a very little greater than the fifth of an ordinary keyed instrument. The E. T. third is four semitones; so that the perfect third is a little less than the third of an ordinary keyed instrument.

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* See note at the end of Chap. III., and Appendix.

The deviation from E. T. values may be called *Departure.* Thus the perfect fifth is said to have the *departure* $\frac{1}{51}$ upwards from E. T. Deviation from exact concords may be called *Error*; thus the *error* of the E. T. fifth is $\frac{1}{51}$ down. So the *departure* of the exact third is $\frac{1}{7.3}$ down; and the *error* of the E. T. third is $\frac{1}{7.3}$ up.

The practical effect of these deviations is that the E. T. fifth has to be made about one beat per second flat in the lower part of the treble. The beats of a simple third are generally difficult to distinguish clearly as beats; their number in the same region is about ten per second. The beats of the first combination tones of an E. T. major triad in the same region are about five per second*. The error due to thirds is considerable in the equal temperament; that due to fifths is small in comparison.

To obtain some practical idea of the difference in the sound

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* For tuning equal temperament with accuracy the following table may be employed, c′=264 :—

Beats per minute of flat fifths.

c′—g′	53·4	f♯—c″♯	75·6
c′♯—g′♯	57	g′—d″	80·4
d′—a′	60·3	a′♭—e″♭	85·2
e′♭—b′♭	64·2	a′—e″	90
e′—b′	67·5	b′♭—f″	95·4
f′—c″	71·6	b′—f♯	101·4

Proceed in order of fifths, thus; c′—g′—d″, then octave down d″—d′, and so on.

Mr. Ellis has given a useful practical rule, which is more manageable than the above, and does not err in its results by much more than the hundredth part of a semitone. It is—make all the fifths which lie entirely within the octave c′ c″ beat once per second; and make those which have their upper notes above c″ beat three times in two seconds. Keeping the fifth f′—c″ to the last, it should beat once in between one and two seconds. See Ellis's Helmholtz, p. 785. This is a perfectly practicable rule, and tuners ought to be instructed in the use of it. There are few tuners who can produce a tolerable equal temperament, of equal temperament chords and perfect chords, the simplest thing to do is perhaps to take an ordinary harmonium and tune two chords perfect on it. One is scarcely enough for comparison. To tune the triad of C major first raise the G a very little, by scraping the end of the reed, till the fifth C—G is dead in tune. Then flatten the third E, by scraping the shank, until the triad C—E—G is dead in tune. (Whenever a third is to be tuned perfect, a perfect fifth ought to be made, and the third tuned in the middle of the triad.) Then flatten F till F—C is perfect, and A till F—A—C is perfect. The notes used are easily restored by tuning to their octaves. Any small sharp chisel will do to tune with; a thin and narrow strip of steel or stiff card is useful to place under the reed so as to hold it fast. The pure chords obtained by the above process offer a remarkable contrast to any other chords on the instrument.

This experiment is perhaps the most striking practical mode of shewing that chords formed by the notes ordinarily in use are much inferior in excellence to chords which are in perfect tune.

CHAPTER III.

SYSTEM OF PERFECT FIFTHS. PYTHAGOREAN SYSTEM.

CERTAIN intervals produced by tuning perfect fifths bear the name Pythagorean. The tuning of exact fifths on the harmonium is very easy and certain; and it is recommended that the observations about to be made be thus verified experimentally.

Tune the following set of twelve exact fifths or fourths, g♭—d♭—a♭—e♭—b♭—f—c—g—d—a—e—b—f♯. Then f♯ will be higher than g♭ by a small interval called the Pythagorean comma.

For this purpose it is convenient to have a harmonium with two sets of reeds. The f♯ can then be tuned on the second set by tuning its b first to the b on the first set. If there be only one set of reeds, the g♭ and f♯ must be taken an octave apart.

Major tone.

c—d is a major tone of the theorists. Then six major tones exceed an octave by the Pythagorean comma.

c—d is arrived at by tuning two fifths up and one octave down; or $2 \times 7\frac{1}{51} - 12 = 2\frac{2}{51}$.

$$\text{Six major tones} \quad = 6 \times 2\frac{2}{51}$$

$$= 12\frac{12}{51},$$

and $\frac{12}{51}$ is the Pythagorean comma.

Pythagorean comma.

We can deduce the value of the Pythagorean comma, or the departure from E. T. of twelve perfect fifths, directly, by supposing that we tune twelve fifths up and seven octaves down.

$$12 \times 7\frac{1}{51} - 7 \times 12 = \frac{12}{51},$$

or a little less than $\frac{1}{4}$ of an E. T. semitone. The accurate value is $12 \times .01955 = .23460$.

(NOTE.—The fifth is accurately $\frac{1}{51.151}$, and we must remember that we use $\frac{1}{51}$ as an approximation only.)

Pythagorean third.

f—a, c—e, g—b, and other thirds arrived at by tuning four fifths upwards, are very sharp, and are called Pythagorean thirds, or sometimes 'dissonant' thirds. Using the chords c — e — g, c — f — a, we hear the disagreeable effect of the sharp third.

Ordinary comma.

The comma is defined as the difference between the Pythagorean third and the perfect third.

The Pythagorean third is $4 \times 7\frac{1}{51} - 2 \times 12 = 4\frac{4}{51}$;

the perfect third is $4 - \frac{1}{7.3}$;

so that the comma becomes

$$\frac{4}{51} + \frac{1}{7.3} = \frac{1}{4.6} \text{ nearly;}$$

or, using the more convenient and accurate decimal values,

$$4 \times .01955 + .13686 = .21506.$$

This number may also be found by the rule at the end of this chapter from the vibration ratio $\left(\frac{81}{80}\right)$ of the ordinary comma.

Apotomè Pythagorica.

The semitone formed by tuning seven fifths is given by $b\flat - b$. It is distinguished by some of the older theorists as Apotomè Pythagorica.

Its magnitude is

$$7 \times 7\frac{1}{51} - 48 = 1\frac{7}{51} \text{ or } 1\frac{1}{7} \text{ nearly.}$$

In decimals, 1.13685.

Pythagorean Semitone.

The semitone formed by tuning five fifths is given by $a - b\flat$. It has been called the Pythagorean semitone.

Its magnitude is $36 - 5 \times 7\frac{1}{51} = 1 - \frac{5}{51}$;

i.e. it is less than the E. T. semitone by about $\frac{1}{10}$ of a semitone*.

In decimals, $1 - .09775$.

Pythagorean Sixth.

The interval $c - a$ is commonly called a Pythagorean sixth.

Its value is $3 \times 7\frac{1}{51} - 12 = 9\frac{3}{51}$;

i. e. it exceeds the E. T. sixth by $\frac{1}{17}$ of a semitone.

---

* The diatonic semitone of theorists is the difference between an exact third and fourth, or $(5 - \cdot 01955) - (4 - \cdot 13686) = 1\cdot 11731 = 1\frac{1}{8\cdot 4}$ nearly. The chromatic semitone is the difference between this and a major tone, $= 2\cdot 03910 - 1\cdot 11731 = 1 - \cdot 07821 = 1 - \frac{1}{13}$ nearly.

Helmholtz's Theorem.

The following theorem has been brought into notice by Helmholtz. In a series of perfect fifths, any two notes eight steps apart determine a major third which is nearly perfect.

Thus if we take an f♯ identical in pitch with g♭, c♯ with d♭, g♯ with a♭, and d♯ with e♭, then d—f♯, a—c♯, e—g♯, b—d♯, are very nearly perfect thirds.

For five octaves less eight fifths

$$=60-8\times 7\frac{1}{51}$$

$$=60-56\frac{8}{51}$$

$$=4-\frac{8}{51};$$

or the third thus obtained is less than the E. T. third by

$$\frac{8}{51} \text{ or } \frac{1}{6.4} \text{ nearly.}$$

But the perfect third is less than the E. T. third by

$$\frac{1}{7.3}.$$

The difference of these numbers is $\frac{1}{51}$ nearly*; or nearly the same as the error of one E. T. fifth, and $\frac{1}{7}$ of the error of the E. T. third. The chords formed with the above thirds

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* In decimals $8\times 01955 = \cdot 15640$

$$\begin{array}{r} \cdot 13686 \\ \hline \cdot 01954. \end{array}$$

This quantity is sometimes called a skhisma.

Note on Relations of Semitones, Comma, and Skhisma.

Diatonic Semitone	=	Pythagorean Semitone	+	Comma
	=	Apotomè Pythagorica	−	Skhisma.
Chromatic Semitone	=	Apotomè Pythagorica	−	Comma.
	=	Pythagorean Semitone	+	Skhisma.
Pythagorean Comma	=	Comma	+	Skhisma.
	=	Apotomè Pythagorica	−	Pythagorean Semitone.

These identities are easily verified by means of the decimal values.

are therefore very nearly perfect; and the experiment enables us to contrast in an effective manner Pythagorean chords with chords of a good quality.

Notation for series of Fifths.

As the f♯, c♯, g♯, and d♯ above mentioned are identical in pitch with g♭, d♭, a♭, and e♭, respectively, it is necessary to adopt some notation to distinguish these notes from those on the right hand of the series of fifths, which are derived from modulation through upward fifths, and differ in pitch from the notes last introduced, though bearing the same names.

The notation employed is as follows, if the whole series be linked by exact fifths, and supposed indefinitely extended according to the same law in both directions. The E. T. names (♭ or ♯), are used indifferently. \\\b — \\f♯ — \\c♯ — \\g♯ — \\d♯ — \\b♭ — \\f — \\c — \\g — \\d — \\a — \\e — \\b — \f♯ — \c♯ — \g♯ — \d♯ — \b♭ — \f — \c — \g — \d — \a — \e — \b — f♯ — c♯ — g♯ — d♯ — b♭ — f — c — g — d — a — e — b — /f♯ — /c♯ — /g♯ — /d♯ — /b♭ — /f — /c — /g — /d — /a — /e — /b — //f♯.

The notes comprised in any one series of twelve fifths from f♯ up to b, all bear the same mark. In the middle there is a series without marks; as we pass to the left we have series with one or more marks of depression (\); as we pass to the right, with one or more marks of elevation (/). The ordinary names (♯ or ♭) are used indifferently; the notation alone marks the position in the series of fifths.

In each series we have four major thirds such as d—f♯, a—c♯, e—g♯, b—d♯.

In each pair of adjoining series we have eight major thirds such as g♭—\b♭, d♭—\f, a♭—\c, e♭—\g, b♭—\d, f—\a, c—\e, g—\b.

We may embody this in the rule, that the four notes to the right of any series form thirds with the four notes to the left; but all other thirds lie in adjoining series. The four notes to the right, which have their thirds in their own series, are the letters of the word *bead*, which may be employed for the purpose of remembering them.

This notation can be used in the musical staff; and something of the kind is essential when thirds formed by eight fifths are employed.

Example. 

System of 53.

It is easy to see that the division of the octave into fifty-three equal intervals has very nearly perfect fifths, without going into any general theory. For taking thirty-one units for the fifth, twelve fifths make 372 units,

and seven octaves $= 7 \times 53 = 371$ units;

or twelve fifths exceed seven octaves by one unit;

and one unit is $\frac{12}{53}$ of an E. T. semitone.

But the excess of twelve fifths over seven octaves is the departure of twelve fifths from E. T.

(for in E. T. twelve fifths = seven octaves).

Hence the departure of twelve fifths $= \frac{12}{53}$;

whence departure of one fifth $= \frac{1}{53}$; a simple and elegant result.

But departure of perfect fifth $= \frac{1}{51}$ nearly.

$$\text{Therefore error of fifth of } 53 = \frac{1}{51} - \frac{1}{53}$$
$$= \frac{2}{2703}$$
$$= \frac{1}{1352} \text{ nearly,}$$

or less than the one thousandth of a semitone; an inappreciable error.

We may here point out that the diatonic semitone and apotomè Pythagorica are both closely represented by five units of the system of 53, and the chromatic semitone and Pythagorean semitone by four units of the same system. For more extensive comparisons of this kind see Stainer and Barrett's 'Dictionary of Musical Terms,' p. 423.

In 'Hopkins on the Organ,' 2nd edition, p. 160, there are some small inaccuracies on this subject which it may be as well to correct. The statement about 'Tempering,' p. 161, will be alluded to in connection with the mean-tone system. (1) The comma is identified with the 53rd part of an octave. This is not correct. Dividing 12 by ·21506 (the value of a comma), we find that 55·8 commas very nearly make an octave. (2) 'The successive sounds of the diatonic scale have, by the aid of these commas, been shown to be separated by intervals of the following "sizes" or comparative dimensions.' Then the scale is set out, with the major tone represented by nine units, the minor tone by eight, and the diatonic semitone by five. This is the scale of the system of 53, as is easily seen by counting up the intervals. But it is not the diatonic scale, only an approximation to it, and the difference of the thirds and sixths in the two is very sensible. The diatonic scale is such that thirds fifths and sixths are perfect. The scale of 53 coincides very closely with that of a system of perfect fifths, but its thirds and sixths are not very close approximations to those of the diatonic scale, though sufficient for many purposes.

In fact, by Helmholtz's theorem, the major third determined by notes eight steps apart, in a series of perfect fifths, is too flat by nearly the same quantity as the equal temperament fifth, or .01954. Although we may frequently neglect this small error, and establish on this neglect practical approximate methods of importance, yet a fundamental exposition in which it is entirely overlooked can only be regarded as erroneous. The state of things in the system of 53 is very nearly the same as in the system of perfect fifths. The exact values are easily calculated.

Those who are not familiar with the properties of logarithms, are referred to the Appendix, on the theory of the Calculation of Intervals.

Note on the Calculation of Intervals.

To transform the logarithms of vibration ratios into E. T. semitones.

The vibration ratio of the octave is 2; the logarithm of 2 is ·3010300; and we admit that E. T. intervals are in effect a system of logarithms such that 12 is the logarithm of the octave, or of 2. Then, since different systems of logarithms can always be transformed one into another by multiplication by a certain factor or modulus, we have only to find the factor which will convert ·3010300 into 12. The simplest proceeding which embodies this process directly and conversely is given in the following rules, which admit of transforming the logarithms of vibration ratios into E. T. semitones, and *vice versâ*, with considerable accuracy and facility. For examples worked out at length, see Proceedings of the Musical Association, 1874–5, p. 7.

Rule I. To find the equivalent of a given vibration ratio in E. T. semitones.

Take the common logarithm of the given ratio; subtract $\frac{1}{300}$ thereof, and call this the first improved value (F. I. V.). From the original logarithm subtract $\frac{1}{300}$ of the first improved value, and $\frac{1}{10000}$ of the first improved value. Multiply the remainder by 40. The result is the interval expressed in E. T. semitones correctly to five places.

Rule II. To find the vibration ratio of an interval given in E. T. semitones.

To the given number add $\frac{1}{300}$ and $\frac{1}{10000}$ of itself. Divide by 40. The result is the logarithm of the ratio required.

For approximate work a simpler and less accurate form is sometimes useful; for this purpose the rules can be modified as follows:—

Approximate Rule I.

To find the equivalent of a given vibration ratio in E. T. semitones, where not more than three places are required to be correct.

Take the common logarithm of the given ratio; subtract $\frac{1}{300}$

thereof, and multiply the remainder by 40. The result is the interval in E. T. semitones correctly to three places.

Example. To find the approximate value of a perfect fifth, the vibration ratio of which is $\frac{3}{2}$, and of a perfect third, the vibration ratio of which is $\frac{4}{5}$:

$$\begin{array}{lr} \log. 3 = \cdot 47712 \\ \log. 2 = \cdot 30103 \\ \hline \log. \frac{3}{2} = \cdot 17609 \\ \frac{1}{300} = \cdot 00059 \\ \hline \cdot 17550 \\ 40 \\ \hline 7 \cdot 02|000 \end{array} \qquad \begin{array}{lr} \log. 5 = \cdot 69897 \\ \log. 4 = \cdot 60206 \\ \hline \log. \frac{5}{4} = \cdot 09601 \\ \frac{1}{300} = \cdot 00032 \\ \hline \cdot 09659 \\ 40 \\ \hline 386360 \\ \hline \text{or,}\ 4 - \cdot 136|40 \end{array}$$

The correct values are 7·0195500, and 4 — ·136863.

Approximate Rule II.

To find the vibration ratio of an interval given in E. T. semitones, where not more than three places are required correct.

To the given number add $\frac{1}{300}$ of itself. Divide by 40. The result is the logarithm of the ratio required.

Example. The E. T. third is 4 semitones.

$$\begin{array}{r} 4 \cdot 000 \\ \frac{1}{300} = \quad \cdot 013 \\ \hline 40)4 \cdot 013 \\ \hline \cdot 1003 = \log. 1 \cdot 259 \end{array}$$

The correct value is 1·259921.

N.B. The approximate rules are insufficient for the calculation of beats.

Note on the Calculation of Beats.

It is frequently necessary for the solution of problems in tuning to calculate the number of beats per second or minute made by imperfect unisons, fifths, or thirds.

The following are principles which we shall admit for purposes of calculation. For their more detailed treatment reference is made to

Ellis's translation of Helmholtz; or to the elementary works of Tyndall and Sedley Taylor on the theory of sound.

Musical notes reach us as periodic impulses of the air. The pitch of the note depends on its number of vibrations per second; and the interval between two notes depends on the ratio of the vibration numbers. A note may be regarded as generally containing many notes of the simplest kind, frequently called simple tones. The lowest of these is that with which we identify the compound note; it is called the fundamental; the remainder are called harmonics; and the forms they can take are as follows, the successive tones which make up the note being enumerated in a sequence which is called their order.

Interval from Fundamental in semitones.	Order and proportionate vibration number.	Name.
	1	Fundamental.
12.00000	2	Octave.
19.01955	3	Twelfth.
24.00000	4	Fifteenth or double octave.
28—.13686	5	Tierce (octave tenth).
31.01955	6	Octave twelfth.
34—.31174	7	Harmonic seventh.
36.00000	8	Triple octave.

First Rule of Beats.

When two simple tones are near together in pitch they give rise to alternations in intensity called beats: the number of beats is the difference of the vibration numbers; and the two are said to interfere.

Second Rule of Beats.

When two compound notes form any consonant interval, two of their harmonics coincide in pitch; and if the interval is not exact, the two harmonics coincide very nearly, and give beats according to the first rule.

Examples.—Two notes whose vibration numbers are 32 and 34 per second are sounded together; resulting beats of fundamentals are two per second. (Rule 1.)

In the same two notes, the vibration numbers of the twelfths are 96 and 102, and the beats due to the twelfths are six per second (Rule 2.)

This is easily verified by sounding simultaneously the lowest c and c♯ of a Pedal Bourdon on the organ. The Bourdon note contains no octave; and the two classes of beats above mentioned combine to produce an effect like U — u — u — U — u — u in every second, where U is the beat of the fundamentals, u of the twelfths.

To find the beats per minute of the equal temperament fifth c′ g′. (c′= 256.)

The octave of g′ will interfere with the twelfth of c′, the two notes being separated by the interval ·01955 of a semitone, by which the equal temperament fifth is flat, since an exact fifth contains 7·01955 semitones.

To find the logarithm of the ratio for the interval ·01955, we proceed by Rule II of the preceding note.

$$\begin{array}{rl} \cdot 0\dot{1}95500 & \\ \cdot 0000652 = & \dfrac{1}{300} \\ 19 = & \dfrac{1}{10000} \\ 40)\overline{\cdot 0196171} & \\ \cdot 0004904 & \end{array}$$

=logarithm of vibration ratio for ·01955 semitones.

The vibration number of g″ derived from c′ is 768.

log. 768	=	2·8853613
log. ratio	=	·0004904
log. vibration number of tempered g″	=	2·8848709
	=	log. 767·133

The number of beats per second is the difference of the vibration numbers

$$\begin{array}{rl} 768\cdot 000 & \\ 767\cdot 133 & \\ \hline \cdot 867 = & \text{number of beats in one second} \\ 60 & \\ \hline 52\cdot 02 = & \text{number of beats in one minute.} \end{array}$$

Rule of Difference Tones.

When two tones sound together a third is produced, whose vibration number is the difference of those of the first two.

Examples—To find the difference tone of the equal temperament major third c′ — e′.

By Rule II. of preceding note we find for the correct logarithm of the ratio of the tempered third,

$$\begin{array}{lr} & \cdot1003433 \\ \text{also} \quad \log. & \underline{256 = 2\cdot4082400} \\ & 2\cdot5085833 = \log.\ 322\cdot54 \end{array}$$

which is the vibration number of the tempered e′.

Whence, for the difference tone— $\begin{array}{r} 322\cdot54 \\ \underline{256\cdot00} \\ 66\cdot54 \end{array}$

64 would be C; this is about half a semitone sharper.

Again, to find the difference tone of the equal temperament minor third, e′ — g′.

$$\begin{array}{lr} g' \text{ is} & 383.57 \\ e' \text{ is} & \underline{322.54} \\ \text{difference tone} & 61.03 \end{array}$$

Beats of Difference Tones.

Difference tones which lie near each other in pitch interfere and cause beats.

Example.—To find the beats of the difference tones of the equal temperament triad c′ — e′ — g′.

By the two last examples the difference tones of the major and minor thirds lie near each other; they are about a semitone apart. Taking the difference of their vibration numbers, we have

$$\begin{array}{r} 66.54 \\ \underline{61.03} \\ 5.51 \end{array}$$

or $5\frac{1}{2}$ beats per second nearly.

CHAPTER IV.

THE GENERALISED KEYBOARD.

IN the enharmonic harmonium exhibited by the writer at the Loan Exhibition of Scientific Instruments, at South Kensington, 1876, there is a keyboard which can be employed with all systems of tuning reducible to successions of uniform fifths; from this property it has been called the generalised keyboard. It will be convenient to consider it in the first instance with reference to perfect fifths; it is actually applied in the instrument in question to the division of the octave into fifty-three equal intervals, which has just been shown to admit of practical identification with a system of perfect fifths.

This keyboard is arranged in a symmetrical manner, so that notes occupying the same relative position always make the same interval with each other. The requisite minuteness is secured by providing two separate indications of the position of each note, the one referring to its position in the E. T. scale, the other to its departure from the E. T. position.

As to the position in the E. T. scale. Suppose the broad ends of the white keys of the piano to be removed; the distance of the octave from left to right is then occupied by twelve keys of equal breadth, seven of which are white, and five black. This is the fundamental division of the new keyboard on any horizontal line. The order of black and white is the same as usual.

But we have also to express departure from any one E. T. system; and this is done by placing the notes at different distances up and down in these divisions. Apply this to the series of exact fifths, starting from c.

$$c-g \text{ is } 7\frac{1}{51}:$$

7 corresponds to the E. T. g, and would be denoted by a position in the g division on a level with c.

But our note is $\frac{1}{51}$ higher.

In the keyboard itself this is denoted by placing the g key $\frac{1}{4}$ of an inch further back and $\frac{1}{12}$ of an inch higher than the c. Similarly d departs from E. T. by twice as much; it is placed $\frac{1}{2}$ an inch back and $\frac{1}{6}$ of an inch higher than c, and so on; every note determined by an exact fifth being placed $\frac{1}{4}$ of an inch further back and $\frac{1}{12}$ of an inch higher than the note which immediately precedes it in the series of fifths.

Thus after twelve fifths, when we come to /c, we find it displaced three inches back, and one inch upwards; a position which admits of its being represented by a key placed behind and above the c key from which we started. The general nature of the arrangement will be best gathered from the two accompanying illustrations. The first is an arrangement of the notes of Gen. T. Perronet Thompson's Enharmonic Organ, in a symmetrical form according to the above principle. The instance is selected as being of historical interest. Each vertical step from dot to dot may be taken for present purposes to represent the departure of an exact fifth, or $\frac{1}{51}$ E. T. S. nearly. Two notes are missing from the complete scheme, b and \\d; their importance is well seen. The second diagram represents a small portion of the generalised keyboard itself. It will be desirable here to fix in the mind the conception of the latter as constituting a mechanical means by which an endless series of uniform fifths can be controlled.

But the most important practical point about the keyboard arises from its symmetry; that is to say, from the fact that every key is surrounded by the same definite arrangement of keys, and that a pair of keys in a given relative position corresponds always to the same interval. From this it follows that any passage, chord, or combination of any kind, has exactly the same form under the fingers in whatever key it is played. And more than this, a common chord for instance has always exactly the same form, no matter what view be taken of its key relationship. Some simplification of this kind is a necessity if these complex phenomena are to be brought within the reach of persons of average ability; and with this particular simplification, the child or the beginner finds the work reduced to the acquirement of one thing, where twelve have to be learnt on the ordinary keyboard.

Hitherto it has been assumed that we were dealing with perfect fifths; but it is clear that this is not a necessary condition; the keyboard will serve to represent any continuous series of fifths which keeps on departing from the E. T., even though they should be flatter than the E. T. fifths. As an illustration of the generality of its properties we will consider its application to the mean-tone system. But as this system is one of the most important with which we have to deal, we will first devote some space to an account of its history and properties. We shall recur later to the properties of systems of perfect and approximately perfect fifths.

I.

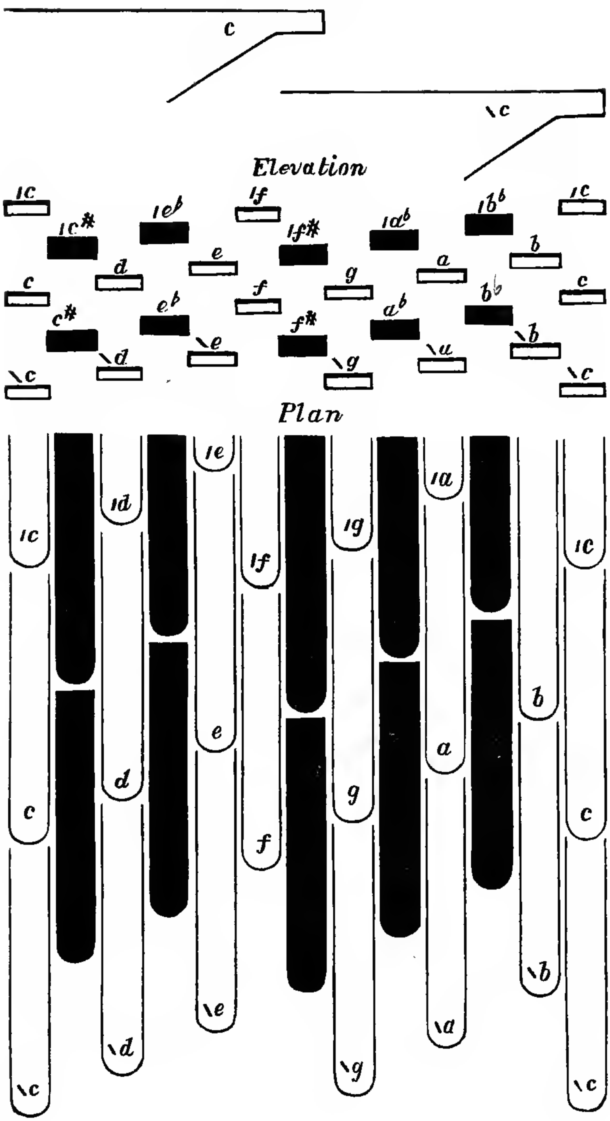
Symmetrical Arrangement of the Notes of Thompson's Enharmonic Organ.

The subscripts $_{1}$, $_{2}$, $_{3}$ refer to its three keyboards.

12.	/c$_1$	.	.	.	.	.	.	.	.	.	.	.	/c$_1$
11.	.	.	.	.	.	/f$_1$	.	.	.	.	.	.	.
10.	.	.	.	.	.	.	.	.	.	.	/b♭$_1$	.	.
9.	.	.	.	/e♭$_1$	.	.	.	.	.	.	.	.	.
8.	.	.	.	.	.	.	.	.	/a♭$_1$	.	.	.	.
7.	.	/♯c$_1$	.	.	.	.	.	.	.	.	.	.	.
6.	.	.	.	.	.	.	/f♯$_1$	.	.	.	.	.	.
5.													
4.	.	.	.	.	e$_2$	.	.	.	.	.	.	.	.
3.	.	.	.	.	.	.	.	.	.	a$_{1,2}$	.	.	.
2.	.	.	d$_{1,3}$	.	.	.	.	.	.	.	.	.	.
1.	.	.	.	.	.	.	.	g$_{1,2,3}$	.	.	.	.	.
12.	c$_{1,2,3}$	.	.	.	.	.	.	.	.	.	.	.	c$_{1,2,3}$
11.	.	.	.	.	.	f$_{1,2,3}$	.	.	.	.	.	.	.
10.	.	.	.	.	.	.	.	.	.	.	b♭$_{1,3}$	.	.
9.	.	.	.	e♭$_{1,3}$	.	.	.	.	.	.	.	.	.
8.	.	.	.	.	.	.	.	.	a♭$_3$	.	.	.	.
7.	.	c♯$_2$	.	.	.	.	.	.	.	.	.	.	.
6.	.	.	.	.	.	.	f♯$_{1,2}$	.	.	.	.	.	.
5.	.	.	.	.	.	.	.	.	.	.	.	\b$_{1,2,3}$	.
4.	.	.	.	.	\e$_{1,2,3}$	.	.	.	.	.	.	.	.
3.	.	.	.	.	.	.	.	.	.	\a$_{1,2,3}$	.	.	.
2.	.	.	\d$_{1,2,3}$	.	.	.	.	.	.	.	.	.	.
1.	.	.	.	.	.	.	.	\g$_{1,2,3}$	.	.	.	.	.
12.	\c$_3$	.	.	.	.	.	.	.	.	.	.	.	\c$_3$
11.	.	.	.	.	.	\f$_3$	.	.	.	.	.	.	.
10.	.	.	.	.	.	.	.	.	.	.	\b♭$_2$	.	.
9.	.	.	.	\e♭$_2$	.	.	.	.	.	.	.	.	.
8.	.	.	.	.	.	.	.	.	\a♭$_{2,3}$	.	.	.	.
7.	.	\c♯$_{2,3}$	.	.	.	.	.	.	.	.	.	.	.
6.	.	.	.	.	.	.	\f♯$_{2,}$	.	.	.	.	.	.
5.	.	.	.	.	.	.	.	.	.	.	.	\\b$_{2,3}$	.
4.	.	.	.	.	\\e$_3$	.	.	.	.	.	.	.	.
3.	.	.	.	.	.	.	.	.	.	\\a$_3$	.	.	.
2.	.												
1.	.	.	.	.	.	.	.	\\g$_2$	.	.	.	.	.
12.	\\c$_2$	.	.	.	.	.	.	.	.	.	.	.	\\c$_2$
11.	.	.	.	.	.	\\f$_3$	.	.	.	.	.	.	.
10.	.	.	.	.	.	.	.	.	.	.	\\b♭$_2$	.	.
9.	.	.	.	\\e♭$_2$	.	.	.	.	.	.	.	.	.
8.	.	.	.	.	.	.	.	.	\\a♭$_2$	.	.	.	.
7.	.	\\c♯$_3$	.	.	.	.	.	.	.	.	.	.	.

II.

Section.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CHAPTER V.

MEAN-TONE SYSTEM. OLD UNEQUAL TEMPERAMENT.

WE saw that four exact fifths upwards lead to a third (c—e), a comma sharper than the perfect third (by definition of comma). If then we make each of the four fifths $\frac{1}{4}$ of a comma flat, the resulting third is depressed a whole comma, and coincides with the perfect third. This is the rule of the mean-tone system; the fifths are all $\frac{1}{4}$ of a comma flat. It is called the mean-tone system because its tone is the arithmetic mean between the major and minor tones of the diatonic scale, or half a major third. The historical interest of this system is very great. It can be traced back with certainty as far as two Italian authors of the sixteenth century, Zarlino and Salinas, and some claim for it a much higher antiquity. From this time it spread slowly, and about 1700 was in universal use. The early development has many points which are historically obscure, but one of special interest, if anything could be found out about it, is the connection of this system with our present musical notation. We shall see that in this system the distinction between such notes as c♯ and d♭ is true and essential *; so that in the earliest times of what we can call modern music, we find a system of notes in use with a notation which exactly represents its properties. It is impossible to avoid the surmise that the two may have

* We have seen that this distinction is false as applied to systems of the type of the system of perfect fifths; for such a note as c♯ has in those cases two forms, the one of which is practically identical with d♭, and the other higher in pitch.

had a common origin; and perhaps it would not be difficult to make out a plausible case for Guido d' Arezzo, to whom both have been ascribed; but the evidence is too defective to build much upon.

The historical account of the introduction of 'Tempering' in Hopkins on the Organ, p. 161, is not quite correct. He does not allude to the mean-tone system at all. But it is described clearly by all the principal writers, and there can be no doubt that it was the usual form of the old unequal temperament*.

The principal interest of this question for us is the fact that it was the system employed by Handel and his contemporaries under the form known as 'the old unequal temperament,' and that it kept its ground in this country until within the last few years. There are still organs remaining which are tuned in this manner†. Indeed we may say with considerable accuracy that this system was the language of music for nearly two hundred years.

There can be no doubt that, with the musicians of Handel's time, the good keys of the old unequal temperament, i.e. the mean-tone system, formed the ideal of the best tuning

* The following note on the origin of the mean-tone system is quoted from Smith's Harmonics, p. 37.

'Salinas tells us, that when he was at Rome, he found the musicians used a temperament there, though they understood not the reason and true measure of it, till he first discovered it, and Zarlino published it soon after. . . . .

'After his return into Spain, Salinas applied himself to the Latin and Greek languages, and caused all the ancient musicians to be read to him, for he was blind; and in 1577 he published his learned work upon music of all sorts; where treating of three different temperaments of a system, he prefers the diminution of the fifth by a quarter of a comma to the other two. . . . .

'Dechales says, that Guido Aretinus was the inventor of that temperament. . . . . But that opinion wants confirmation. . . . .'

In Smith's own discussions he generally employs the expression 'system of mean tones,' in speaking of this temperament.

† Instances are, the organ at St. George's Chapel, Windsor; and the magnificent instrument in Turvey Church.

attainable. The proof of this is to be found in the fact that Handel took the trouble to employ an arrangement, by which the range of good keys available on the ordinary board with this system could be somewhat extended. It is well known that he presented to the Foundling Chapel an organ possessing additional keys for this purpose. The organ at the Temple Church in its original state, as built by Father Smith, possessed a similar arrangement. The principle will be explained subsequently. Here it will be sufficient to cite the following description of the instrument given by Hopkins*, the well-known organist at the Temple, in his work on the organ:—

'The fine organ in the Temple Church was built by Father Smith in 1688. It presents a great peculiarity in regard to the number of sounds which it contains in the octave. Most organs have only twelve in that compass, but this has fourteen: that is, in addition to the common number of semitones, it possesses an A♭ and D♯, quite distinct from the notes G♯ and E♭. The general temperament of the instrument is the same as that of most English instruments—unequal;—but the real beauty of the quarter tones is discoverable by playing in the keys† of E♭ and A♭, where in consequence of the thirds being so true we have that perfection that cannot be met with in common organs. It gives a peculiar brilliancy also to the keys of A and E in 3 or 4 sharps ‡. These quarter tones are produced by the ordinary G♯ and E♭ keys being divided crossways in the middle; the back halves of which rise as much above the front portions, as do the latter above the naturals.'

These extra keys have long been removed.

The great objection to this system was, that the circle of fifths deviated widely from the equal temperament, and consequently did not meet at the ends §, and the chords which

* Hopkins, On the Organ, 1st ed., 1855, p. 448; 2nd ed., 1870, p. 452.

† This passage is slightly modified so as to make the sense clear.

‡ Reference to Christian Remembrancer, 1833; from which the greater part of the above appears to be quoted.

§ For details see next chapter.

involved notes taken from the two extremities of the circle were exceedingly bad, their effect being commonly known as 'the wolf.' This was undoubtedly the only objection felt to the system by the musicians of Handel's day. The objection is certainly a good one, so long as efficient means for avoiding the wolf are not forthcoming; and no stronger testimony could be produced to the superiority of the good portion of the system over the equal temperament, in the opinion of Handel and his contemporaries, than the fact that they seem, with few exceptions, to have preferred facing the wolf of the unequal temperament to abandoning all the excellences of that system in favour of the equal temperament.

The history of Bach in connection with this subject is of great importance; unfortunately little can be ascertained about it. A few points have however come down to us.

(1) As to clavichord, harpsichord, and piano.

It appears that Bach possessed a clavichord and a harpsichord. The latter was probably the harpsichord with two manuals and pedal, for which the set of six sonatas, in the first volume of the Peters edition of the organ works, was written *. But the clavichord possesses more interest in relation to Bach. The first peculiarity was that, as he used it, it was not 'gebunden.' This necessitates an explanation. The clavichord was an instrument in which a brass wedge attached to the rising end of the key struck the string and at the same time performed the office of a bridge, stopping off the requisite length of string. Advantage was taken of this in early times to produce two or more different notes from one string, and instruments arranged in this manner were called 'gebunden,' or 'tied.' When this was the case,—if c and c♯, for instance, were made on the same string,—the interval between them was determined once for all by the position of the brass wedges †. From this arrangement Bach freed

* Hilgenfeldt, p. 135.

† Ibid. p. 36. This is the only intelligible explanation of the word 'gebunden' that the writer has ever seen. The word is used in Forkel, but no explanation is given, and the passage is unintelligible.

himself; and we find this stated with an emphasis which is now unintelligible without explanation.

It is also stated that the harpsichord and clavichord were the instruments upon which Bach used the equal temperament; and that he always tuned them himself.

Bach's favourite instrument was the clavichord. He considered it the best instrument for the house, and for study. To appreciate this fully we must obtain some idea of the effect of the clavichord. It is described as very soft and expressive in tone: this last quality is said to have been that which so endeared the instrument to Bach*. It was said that it could hardly be heard at any distance. It is much to be wished that some such instrument existed for purposes of study in the present day. The qualities of the clavichord are important with reference to Bach's estimate of the equal temperament.

Through the kindness of Mr. Carl Engel the writer was permitted to examine and play upon a clavichord in excellent order, with Mr. Parratt's assistance; it was submitted to as thorough an examination as seemed necessary for making out its peculiarities. The best tone was produced by a light but decided pressure of the finger; so long as the touch was kept light enough to get a good singing tone, the intensity was exceedingly faint; it seems doubtful whether it would be audible through the least buzz of conversation. With this singing tone the pitch of the notes was fairly constant; but the intensity was far too faint to hear anything of the quality of the chords; and the equal-temperament error certainly could not be objected to on this instrument so far. But further, when any considerable energy of hand was employed, the effect was far from what was expected. The quality deteriorated and the pitch rose considerably when

According to Hilgenfeldt there were generally six strings per octave in the treble, and four in the bass. At p. 37 we find it stated that these instruments were arranged according to the 'Zarlino' sche Temperatur,' or mean-tone system, 'as were all others at that time.'

* Hilgenfeldt, p. 43. In the following passage 'Flügel' means 'harpsichord,' not 'grand pianoforte.' See also note, p. 42.

the pressure passed a certain amount. This variation of the pitch arose from the stretching of the string directly caused by the extra pressure; and its amount was so considerable that it was impossible to regard the instrument as being really anything definite in the way of pitch, when any considerable amount of energy was used. A delicate and beautiful expression was certainly obtainable from the soft tone, but in leading out a subject, or anything which called for a noticeable emphasis, the extra pressure caused a rise in pitch which might amount to a third of a semitone, or more than half a semitone if any considerable pressure were used. On an instrument of this kind, while the wolf of the old unequal temperament would still be offensive, the errors and variations of the instrument itself are so great in comparison to the errors of the equal temperament, that it would seem impossible to get any substantial advantage by seeking for any better method of tuning.

Now it is occasionally said, 'Bach preferred the equal temperament'; and his authority is cited against any attempt to introduce other arrangements. But if it be the case that his favourite instrument was such as we have described it with respect to force and accuracy, it cannot be regarded as any representation of our modern instruments. In particular, any one accustomed to the varying pitch of this instrument must have had an ear not to be shocked by small deviations, and cannot have had the intense feeling for equal-temperament intervals which is characteristic of musicians brought up at the modern piano. The account of Bach's habit of playing on the unequally tempered organ in its worst keys to annoy Silbermann, to which further allusion will be made, confirms this view. This is the same condition of ear with respect to melodic intervals which might be expected to be attained according to the method indicated in the sentence from Kirnberger prefixed to this book; a condition which might well admit of a power of appreciating the distinctions of different systems, and a reference for correctness to the harmonies, instead of to an arbitrary melodic standard.

So far as Bach's clavier music is concerned therefore, the appeal to his authority in favour of the equal temperament falls to the ground. The argument is unfounded in other respects. Bach compared the equal temperament with the defective mean-tone system on the ordinary keyboard, and with nothing else. His objection was to the wolf, and cannot be counted as of force against arrangements in which the wolf does not exist.

It is doubtful whether the title 'wohltemperirte Clavier,' as applied to the 48 preludes and fugues, corresponds to the mature intention of Bach himself. The two parts were composed at different times, as independent works. The second part was regarded as the more important work of the two, and this did not bear the above title under Bach's hand, nor when first published in 1799, nearly fifty years after his death*. The first part was then for the first time printed together with the second, and the title thus got carried over. But it is probable that Bach did not intend the first part to be published at all; and wrote the second later in life to take its place†. The title of the first part is, in the original MSS., 'Das wohltemperirte Clavier, oder Praeludia and Fuga durch alle Tone und Semitonia,' &c. It bears the date 1722. The original title of the second part is—'xxiv Preludien und Fugen durch alle Ton-Arten sowohl mit der grossen als kleinen Terz; verfertiget von Johann Sebastian Bach.' Hilgenfeldt's autograph of the second part bears the date 1740‡.

The pianoforte was developed by Silbermann in Bach's last years. There is evidence that when Bach first became acquainted with it he disliked it. And although we know that he occasionally played upon Silbermann's pianos late in life§, yet we have no evidence as to how they were tuned, or that Bach ever recommended them for study. We know only that Silbermann continued to tune the organ according to the unequal temperament.

In fact it appears that Bach's clavier compositions were

* Hilgenfeldt, p. 85. † Ibid. p. 73. ‡ Ibid. p. 123.

§ There is a well-known story of Frederick the Great taking Bach round to play on all the new Silbermann Fortepianos in the palace at Potsdam. Forkel, p. 10.

regarded both by himself and others as specially dedicated to the clavichord; and when the pianoforte first came into general use, these compositions were forgotten*. It was an idea, excellent no doubt, but belonging to a later period, to take them and apply them to the piano. The step is a great one, but it is not one that Bach himself contemplated. If this is done, his authority cannot be adduced as a reason why further steps in the direction of improvement should not be taken, if we can find such.

(2) The question as to Bach's point of view of the temperament of the organ is much more difficult than is supposed. There is no direct evidence that Bach ever played upon an organ tuned according to the equal temperament. There is evidence to show that he thought the unequal temperament abominable, as anybody would who played as freely as he did; and that he expressed himself very strongly on the matter to Silbermann, who nevertheless continued to tune the organs unequally. There is a well-known story, how, when Silbermann came to listen, Bach would strike up in A♭ as soon as he saw him, saying, 'you tune the organ as you please, and I play as I please.' This must have been late in Bach's life; Silbermann was not likely to have attended often unless the performer's reputation was formed†.

The best evidence, however, is that of Bach's compositions for the organ. There is not a single organ composition of Bach's published in the key of A♭, or any more remote key. There is one in F minor. Compare this with the keys in which his clavier works are written. The comparison furnishes an overwhelming presumption that there was some potent cause excluding the more remote keys.

If therefore it is said on Bach's authority that his organ compositions ought to be played on the equal temperament, it may be answered, that there is no evidence that he played them so himself. But it must always be admitted that they should be played without the 'wolf'; that is all that Bach's authority can be adduced for with certainty.

* In the last third of the eighteenth century. Hilgenfeldt, p. 44.

† Hopkins on the Organ, p. 176.

It is not possible to say with any exactness when the change of organ-tuning to equal temperament took place in Germany. There are considerations which render it probable that unequal temperament still existed in the time of Mozart; and it is possibly owing to this cause that we possess no true organ compositions by this greatest organ-player of his day, except the two gigantic fantasias for a mechanical organ, which are best known as pianoforte duets. As these are both in F minor, it is probable that the instrument for which they were written was tuned to the equal temperament.

All systems which involved 'wolf' have practically disappeared. We shall now discuss the applications of the mean-tone system, and endeavour to show how it can be employed in such a way as to obtain everywhere the excellence formerly peculiar to a few keys, every inequality which gave rise to the wolf being got rid of.

Analysis of the number of Bach's published Organ Compositions in the different Major and Minor Keys, excluding the Chorales.

Major Keys.

E♭	B♭	F	C	G	D	A
2	1	3	13	7	2	1

Minor Keys.

F	C	G	D	A	E	B
1	8	5	7	5	4	2

CHAPTER VI.

APPLICATIONS OF THE MEAN-TONE SYSTEM.

WE have seen that, in the mean-tone system, a series of fifths is tuned according to this rule;—All the fifths are a quarter of a comma flat, the thirds formed by four fifths up being perfect; (for the third formed by four fifths up is a comma sharp if the fifths are exact, by definition of comma). We will first consider the condition of things on the ordinary keyboard when this system was employed, as in the old unequal temperament, pointing out the nature of the various contrivances that have been employed with a view to extend the range of the system in this connection; and then show how the problem of its employment is solved by the generalised keyboard.

To bring clearly before the eye a representation of the arrangement of the mean-tone system as an unequal temperament, we may adopt a method depending on the same principle as that which we employed in the case of the system of perfect fifths. We will arrange the notes first from right to left in order as a series of fifths, and at the same time exhibit the deviation or departure from E. T which accrues as we pass along the series, by displacement downwards for flattening and upwards for sharpening. The notes placed on the twelve numbered lines represent those which existed in the ordinary twelve-keyed unequal temperament: the a♭ and d♯ at the two extremities represent the notes given by the additional keys in the Temple organ. Having thus obtained the departures of the notes up and down, we may now re-arrange them in the order of the scale, and shall thus obtain a graphical representation of the intervals commanded by this arrangement. See Diagrams III and IV.

III.

	(a♭)	.	.	.	.	.	.		.	.	.	.	.	.
1	.	e♭	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	b♭	.	.	.	.		.	.	.	.	.	.
3	.	.	.	f	.	.	.	.	.		.	.	.	.
4	.	.	.	.	c	.	.		.	.	.	.	.	
5	.	.	.	.	.	g	.		.	.	.	.	.	.
6	.	.	.	.	.	.	d	.	.	.	.	.	.	.
7	.	.	.	.	.	.		a	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	e	.	.	.	.	.
9	.	.	.	.	.	.	.		.	b	.	.	.	.
10	.	.	.	.	.	.	.	.	.	.	f♯	.	.	.
11		.	.	.	.	.	.	.	.	.	.	c♯	.	.
12	.	.	.	.	.	.	.	.	.	.	.	.	g♯	.
	.	.	.	.	.	.		.	.	.	.	.	.	(d♯)

IV.

	.	.	.	.	.	.	.	.	(a♭)	.	.	.	.
1	.	.	.	e♭	.	.	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.	.	.	b♭	.	.
3	.	.	.	.	.	f	.	.	.	.	.	.	.
4	c	.	.	.	.	.	.	.	.	.	.	.	c
5	.	.	.	.	.	.	.	g	.	.	.	.	.
6	.	.	d	.	.	.	.	.	.	.	.	.	.
7	.	.	.	.	.	.	.	.	.	a	.	.	.
8	.	.	.	.	e	.	.	.	.	.	.	.	.
9	.	.	.	.	.	.	.	.	.	.	.	b	.
10	.	.	.	.	.	.	f♯	.	.	.	.	.	.
11	.	c♯	.	.	.	.	.	.	.	.	.	.	.
12	.	.	.	.	.	.	.	.	g♯	.	.	.	.
	.	.	.	(d♯)	.	.	.	.	.	.	.	.	.

We have now to investigate the departures of these notes from equal temperament. The amount of this for one fifth is easily found by remembering that each exact major third is equivalent to a proceeding through four fifths of the system, so that the departure of one fifth will be $\frac{1}{4}$ that of the perfect third. The latter is $\frac{1}{7.3}$ downwards; so that the departure of each fifth of the mean-tone system becomes $\frac{1}{29.2}$ downwards*.

The distance between two consecutive dots in a vertical column represents consequently $\frac{1}{29.2}$ E. T. S.; and in Diagram IV the distance between two consecutive dots in a horizontal line represents one E. T. semitone.

We can now examine the nature of the intervals which give rise by their dissonant qualities to the term 'wolf'; and we will suppose that we are dealing with the ordinary twelve-keyed board, i. e. with the unbracketed notes only.

There are two kinds of disturbed intervals; one fifth and four major thirds. The false fifth is that made by the notes marked g♯—e♭ in the diagrams. We see that the g♯ is eleven steps below the e♭. The departure of this fifth is consequently $\frac{11}{29.3}$ downwards, or considerably more than $\frac{1}{3}$ of a semitone lower than E. T. The *error* of this fifth is $\frac{2}{5}$ of a semitone nearly †. This is the worst element of the wolf, on account of the sensitiveness of fifths to tuning.

The four false thirds are those made by the notes marked b—e♭, f♯—b♭, c♯—f, g♯—c in the diagrams. It will be

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* In decimals, $\frac{.13686}{4} = \cdot 034215$.

† $\cdot 034215$

$$\begin{array}{r} \cdot 034215 \\ 11 \\ \hline \cdot 37636 \\ \cdot 01955 \\ \hline \cdot 39591 \end{array}$$ or, .4 nearly.

seen that there are eight steps between the members of each pair, the upper note of the third being above. Consequently the departure of each of these thirds from equal temperament is $\frac{8}{29.3}$ upwards, or rather more than a quarter of a semitone sharper than E. T. thirds, which are too sharp already. The total error of these thirds * is rather more than $\frac{2}{5}$ of a semitone. The false third b – e♭ is a terrible annoyance with this system, as it enters into the keys of E major and minor, which it is hardly possible to keep out of, as the flat keys beyond B♭ are practically unavailable †.

We can now very well understand how great was the gain obtained by introducing the two notes in brackets (Diagram IV), as at the Temple and elsewhere. This particular last-mentioned third was provided for by the (D♯); and the bad fifth E♭ – G♯, which stopped progress so soon in the flat keys, was put one remove further back by the introduction of (A♭); so that the keys of E and E♭ were now fit for use. Considering the small amount of modulation used in old days, it appears intelligible enough that Handel should have been pleased with the arrangement; and the only wonder is that it did not obtain a wider currency.

Besides the application of extra keys to the keyboard other means have been employed for controlling a somewhat extended series of mean tone fifths. The earliest of these seems to have been the arrangement of Smith, who was Master of Trinity College, Cambridge, in the last century, and wrote the well-known treatise on Harmonics, or the Philosophy

$$
\begin{array}{r}
{}^{*}\ \cdot 034215 \\
8 \\
\hline
\cdot 27372 \\
\cdot 13686 \\
\hline
\cdot 41058
\end{array}
$$

† It once fell to the writer's lot to play the Wedding March of Mendelssohn on the organ at Turvey, which is tuned in this manner. The portion in B major produced a horrible effect which will not soon be forgotten, chiefly on account of this false third b – e♭.

of Musical Sounds, 1759. It was an arrangement of stops applied to the harpsichord. The instrument was constructed by Kirkman; but the description of it is so mixed up with a more extended design that it is difficult to say what the arrangements actually were. Assuming that the design of the 'Foreboard,' in fig. II of the separate tract on the changeable harpsichord, represents the actual instrument, it had the following series of fifths :—

D♭ A♭ | E♭ B♭ F C G D A E B F♯ C♯ G♯ | D♯ A♯ E♯ B♯ F𝄪.

Recently Mr. A. J. Ellis, F. R. S., has caused to be constructed a harmonium, in which an extended series of mean-tone fifths is controlled by stops. The resources of the instrument embrace a complete series of keys from seven flats to seven sharps. There is a short notice of the instrument in the Proceedings of the Musical Association, 1874–5, p. 41.

Other instances might be adduced; but the principle does not commend itself as a desirable one. The perpetual interruption of the performance caused by the necessity of changing the stops is a great annoyance; and the method cannot be considered one of practical value, whether stops or change pedals are applied.

We will now consider the application of the generalised keyboard to the mean-tone system. Although a strict adherence to the principle on which our symmetrical arrangements are constructed gives rise to such a distribution as Diagram IV, when the notes are placed in the order of the scale, yet we can by reversing one of our fundamental conventions, reduce this to the same form as that illustrated in Diagram I. We have only to take distances drawn upwards to correspond to flattened pitch, and drawn downwards to sharpened pitch. That is, we represent now a fall by a rise, and a rise by a fall, instead of rise by rise and fall by fall, as in the original application to perfect fifths. Or we may put it thus,—that the distance upwards corresponds simply to advance along an upward series of fifths, without any regard to the question whether the fifths are greater or less than equal temperament fifths, and distance downwards corresponds to a downward series of fifths.

The advantage of thus reducing the mean-tone system, where the departure of the fifths from E. T. is downward, to the same form as the scheme for perfect fifths, where the departure is upward, is twofold. First, when the symmetrical arrangement is embodied in a keyboard, the two things admit of treatment by means of the very same set of keys. Secondly, the form, which the scale and chords of the mean-tone and similar systems assume on the keyboard (Diagram II), is in this case remarkable for facility of execution, and adaptation to the hand.

The sequence of the white unmarked naturals in Diagram II is that which constitutes the mean-tone scale of c, when a series of notes tuned according to the mean-tone rule (fifths quarter of a comma flat), is placed on the keys.

In passing to other scales than that of c, we must first remember that in this system the distinction between such notes as c♯ and d♭ is true and essential. For the major third formed by four fifths up from A is identical with the true major third to A, according to the usage of musicians. So far therefore as thirds and fifths go, we shall dispense with the employment of the notation for position in the series of fifths, and rely upon the distinction between sharps and flats to indicate the key intended to be played.

Since sharps indicate progression through fifths upward, and flats through fifths downward, we have the following rule:—Put the finger up for a sharp, and down for a flat.

Recurring to the symmetrical arrangement of the keyboard, and the fact that for a given rule of tuning the relative position of the notes of a given combination is always the same, we see that all keys have scales of exactly the same form as that of c above described; and the same chord can be reproduced whatever be its key or key relationship with the same form of finger.

In the enharmonic organ built by the writer for the meeting of the Musical Association, May 1, 1875, one of the two stops was tuned according to the mean-tone system. It is called the negative stop on the instrument. The term 'negative' is applied in the general theory to systems which have fifths flatter than equal temperament fifths, i. e. to such as are strictly represented by an arrangement of the form of Diagram

IV. The generalised keyboard has a compass of three octaves, tenor c to c in alt.; and there are forty-eight keys per octave, though only thirty-six were used for the mean-tone stop. The result of this is a range from $d^{\flat\flat}$ to $d^{\sharp\sharp\sharp}$, if we start from the middle of the three c keys as c. Or recurring to the more intelligible system of denoting position in the series of fifths by the notation before described, we have a range from \c to //f; \c becoming $d^{\flat\flat}$ when translated into ordinary notation, and //f becoming $d^{\sharp\sharp\sharp}$. The instrument will be further described in connection with the other stop p. [56].

It is no exaggeration to say that anything can be played on this keyboard, with the mean-tone scales. The movements of the finger required are of the simplest possible character; and the uniformity of the fingering in all keys minimises the necessary study.

To prove the practicability of performance of this kind the writer performed three of Bach's preludes at the meeting of the Musical Association where the enharmonic organ was exhibited, viz. the 1st and 2nd of the first part, and the 9th of the second part. With three octaves only the fugues were not practicable.

But it is not in rapid performance that an arrangement of this kind shews itself specially of value.

The chorale, any massive harmony, not excluding counterpoint, tells well. It is only necessary to remember that we have here the original system, which belongs from the very beginning of modern music onward to our musical notation, to see that by employing it we have the true interpretation of our notation; we have the actual sounds that our notation conveyed to Handel, to all before Bach, and many after him, only cured of the wolf, which was the consequence of their imperfect methods.

It will be unquestionably the case that the modern educated musician will pronounce these notes out of tune. He will not complain of the chords; they are better than equal temperament chords. On examination it will be found that, all the intervals employed being of necessity different from equal temperament intervals, the ear which is highly educated to consider equal temperament intervals right, considers all others wrong; a result by no means strange. But people with good ears, who have not been highly educated as to equal temperament intervals, have no objection to those of the mean-tone system. The semitone is perhaps the best example. The mean-tone semitone is considerably greater than the equal temperament semitone; it is about $1\frac{1}{6}$ E. T. semitones. Eminent modern musicians have said that this semitone was dreadful to them. It was not dreadful to Handel.

The rationale is, that if people who are taught music are taught that one thing is right and another wrong, they will come to believe it. If they were taught the other systems of interest as well as the equal temperament, they would appreciate the excellences of all. By the habit of observing the fine distinctions between them, they would be very much more accurate in their knowledge of any of them separately; and according to the motto from Kirnberger prefixed to this book, other advantages would be likely to accrue as well.

A mean-tone keyboard sufficient for most practical purposes would contain twenty-four keys per octave, and would run from $d^{\flat\flat}$ to $e^{\sharp}$, or from \c to /f in the notation of the series of fifths. This would be practicable and interesting as applied to either organ or harmonium. From experience it is known that the fingering is easy, and the chords are fine.

The tuning of the mean-tone system is understood by organ builders. For the notes c—g—d—a—e, c—e is first made a perfect third, and then the fifths indicated are made equally flat by trial. The group e—b—$f^{\sharp}$—$c^{\sharp}$—$g^{\sharp}$ is then similarly treated, and so on.

CHAPTER VII.

HARMONIC SEVENTH.

HITHERTO we have investigated only chords derived from octave, fifth, and third; but in all the approximate systems to which the generalised keyboard opens a path we can obtain fine effects of a novel character by the introduction of an approximation to the harmonic seventh.

It is well known that if we take a minor seventh such as g—f, and flatten the f by a small interval, we can obtain a seventh, which presents many of the qualities of a consonance, and in which no beats can be heard.

The note f thus determined is the same as the seventh harmonic of a string, whose fundamental is two octaves and the seventh below it; and the vibration ratio of the notes g—f is 4 : 7.

If we compute by Rule I, p. 14, the interval in question, we find for its downward departure from the equal temperament f the value .31174, or a little less than $\frac{1}{3}$ of an equal temperament semitone.

If we consider the system of perfect fifths, we find that the f derived from g by two fifths down has a downward departure due to two perfect fifths, $= \frac{2}{51}$ nearly, or more accurately, $=$ .03910.

If we pass from f downwards through a Pythagorean comma to ↘f (through a circle of twelve perfect fifths), we get a departure from the equal temperament position of .03910 + .23460, the latter number being the value of the Pythagorean comma, and the whole departure = .27370.

Now though this is not quite so great as the required departure of the harmonic seventh, yet it is sufficiently near to it to improve the quality of the interval very much in respect of consonance.

In certain cases then we can use the note \f instead of f in the chord of the dominant seventh on g; and we thereby obtain a chord of very beautiful quality. There exist certain limitations on this use of the approximate harmonic seventh.

Rule. The harmonic seventh on the dominant must never be suspended, so as to form a fourth with the keynote.

For the approximate seventh we can prove this by noticing that the harmonic seventh to dominant g is \f; and c—\f forms a fourth, which is a comma flat nearly. The effect of the flat fourth is bad.

But the rule applies to the harmonic seventh and all its approximations. In ratios this stands as follows:—the ratio of tonic : dominant is 4 : 3; dominant : harmonic seventh as 4 : 7; whence ratio of harmonic seventh of dominant to tonic is 21 : 16, or 63 : 48. But ratio of fourth to tonic is 4 : 3 or 64 : 48; whence this fourth differs from the harmonic seventh to dominant in the ratio 64 : 63, or by more than a comma.

The extreme sharp sixth is susceptible of an improvement in quality by the introduction of the harmonic seventh. An example is contained in the illustration on p. 11.

The mean-tone and allied systems also afford approximations to the harmonic seventh. In these systems the note employed is that derived by ten fifths up from the root; thus the approximate harmonic seventh to g would be e♯ in the ordinary notation, or /f in the notation of the series of fifths. It is convenient to preserve the notation of the series of fifths for this purpose only, with reference to the mean-tone and allied systems.

The departure of one fifth of the mean-tone system being .034215 down*, that of ten fifths becomes .34215, which is a little greater than the required departure for the harmonic seventh.

The harmonic seventh must be used with great caution in the mean-tone and allied systems; its position is so widely removed from equal temperament that its employment in melodic phrases is not generally successful; it is effective in full chords.

* Page 35, note.

CHAPTER VIII.

APPLICATIONS OF THE SYSTEM OF PERFECT FIFTHS AND ALLIED SYSTEMS.

We have seen that the system of 53 is very nearly coincident with that of perfect fifths. There are also a few other systems which may to a certain approximation be treated as practically the same. We will now consider the practical application of these systems.

We shall assume for the present simply that the major third is properly made by eight fifths down, according to Helmholtz's Theorem; and we shall disregard the small error of about $\frac{1}{51}$ of an E. T. semitone (skhisma) which is thus introduced in the thirds.

Though some of the systems of this class are theoretically more perfect than the system of perfect fifths, yet this must always be the most important member of the class in a practical point of view, on account of the ease with which perfect fifths can be tuned.

We will include also in this discussion those practical applications which deal with perfect fifths and thirds, although the establishment of a continuous series of fifths is not aimed at; for these can be conveniently treated by means of the same methods and notation.

Unsymmetrical arrangements.

The work of Mersenne, dated 1636, is an interesting legacy from the time before the mean-tone system was generally established, although it was then well known and spreading. Mersenne exerted immense ingenuity in forming arrangements or systems which should comprise the elements of a number of perfect concords; and he devised keyboards of a most complicated character by which these systems would be controlled. A number of these keyboards are figured in the work. Probably none of them were ever constructed. A detailed account of one of his less complicated systems, having eighteen intervals in the octave, is given at p. 15 and p. 114, of the 'Proceedings of the Musical Association 1874–5.' This may be conveniently summarised in the form of Mr. Ellis's duodenes. Perfect fifths run vertically, major thirds horizontally. The principal key-note was f.

/eb	g	\b	
/ab	c	\e	\g♯
/db	f	\a	\c♯
/gb	bb	\d	\f♯
	eb	\g	\\b

This may be expressed shortly as containing the duodenes of f and \d. Mersenne was very careful about the depressed second of the key,—\g in the principal key of the above; its importance will be discussed presently.

Now directly one examines the possible modulations one is struck by the completeness of the scheme within certain limits; and yet how narrow those limits are when regarded from the point of view of modern music!

The unsymmetrical keyboards would have been terrible things to play. We will now consider their modern representative, Gen. Perronet Thompson's Enharmonic Organ.

In Diagram I, p. 22, a symmetrical arrangement of the notes of this instrument has been already given: the subscripts refer to the three keyboards.

The lowest board has c for its principal key; the second board \e, and the third \d. Each board is tuned so that the ordinary keys give the diatonic scale of the principal key; but on departing from the principal key the notes required to produce concords crop up all over the keyboard in three different shapes, which are called quarrills, flutals, and buttons. Not only has every key a different form, but in all cases where keys or parts of them occur on two or more keyboards the forms are different on the different keyboards. It is astonishing that so much should have been accomplished with this instrument as appears to have been the case.

Thompson also paid great attention to the depressed second of the key.

Key relationship symmetrical arrangements.

The principle of these arrangements is strictly that the form of a chord of given key relationship is the same in every key. But the notes are not all symmetrical, and the same chord may be struck in different forms according to the view which is taken of its key relationship. The form of symmetrical arrangement employed in this treatise may be said to depend on intervals, not on key relationship. The arrangement by intervals includes all the symmetrical properties of the arrangement by key relationship, and much more besides.

The first attempt in this direction was made by H. W. Poole, of South Danvers, Mass., U.S. An account of it is given in 'Silliman's Journal,' July 1867. The keyboard does not appear to have been constructed, but it will be desirable to notice the principle of arrangement. This appears to have been based upon the relations of the different notes of the major and minor scale, but it will be simpler to treat it by the method of intervals.

We may refer the arrangement primarily to two directions, one vertical, in which the principal steps proceed upwards by the apotomè Pythagorica*, and one running upwards to the right at an angle of about 45°, in which the principal notes, or key-note series, proceed by whole tones, in the same manner as the major tones of the generalised keyboard before described; but this law only applies to each row of major tones separately. The different rows are not related as in

* Semitone of seven exact fifths, p. 9.

the generalised keyboard; but according to the rule that the vertical step is the apotomè Pythagorica. These two rules determine the position of the notes called key-notes; i.e. those related to any key-note by fifths, thus—

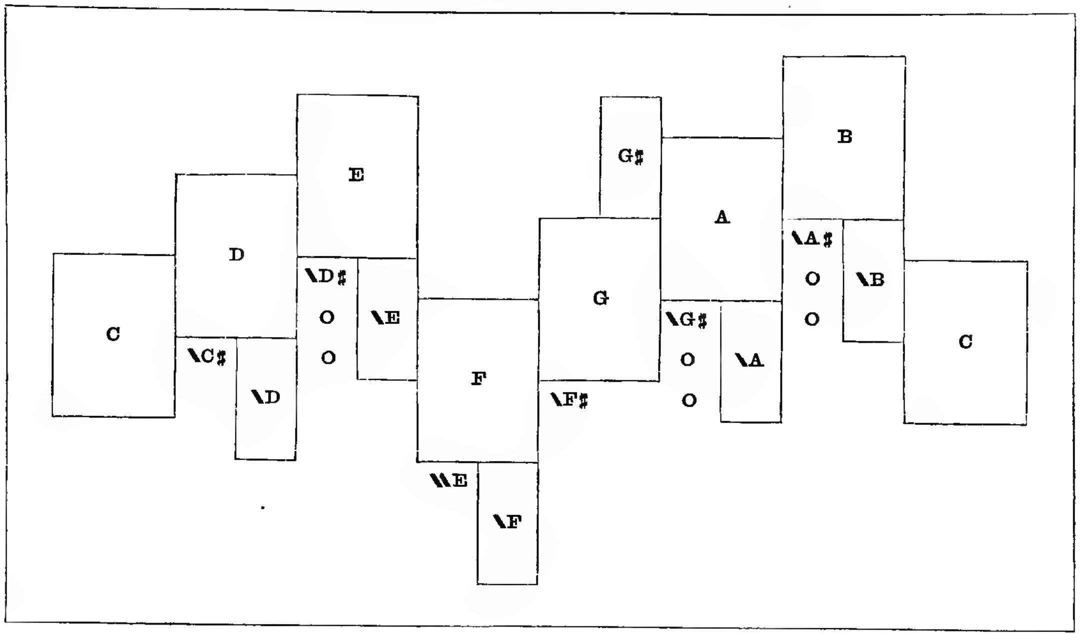
V.

Poole's Keyboard. Colin Brown's Keyboard.
Arrangement of Key-notes.

.	.		.	.	.	B	.
.	.	E	.	.	.	.	.
.	.	.	.	.	A		
.	D	.	.	.	.	.	.
.	.	.	.	G	.		.
C	.	.		.		.	C
.	.	.	F	.	.		.
.	.	.	.	.	.	B♭	.
.	.	E♭	.	.	.	.	.
.	.	.		.	A♭	.	.
.	D♭	.	.	.	.	.	.
.	.	.	.	G♭	.	.	.

These notes are only those which proceed by fifths in each diatonic scale, i.e. F C G D in the key of C. There are besides four series of auxiliaries; thirds to the key-notes, thirds to the thirds, and harmonic sevenths to the key-notes and first set of thirds. We will omit the detailed consideration of the harmonic sevenths, shewing their places only by plain circles. The first set of thirds is provided for by an auxiliary series a comma lower than the principal series; these are placed below their principal notes of the same name in the right-hand half of the division. The left-hand half is occupied by the auxiliaries of the three remaining series, those of the second series of thirds being above and the two harmonic sevenths below. This determines an arrangement for each principal note and its accessories of the same equal temperament derivation, which corresponds somewhat in appearance to a knight's move in chess. In Diagrams VI, VII, VI.

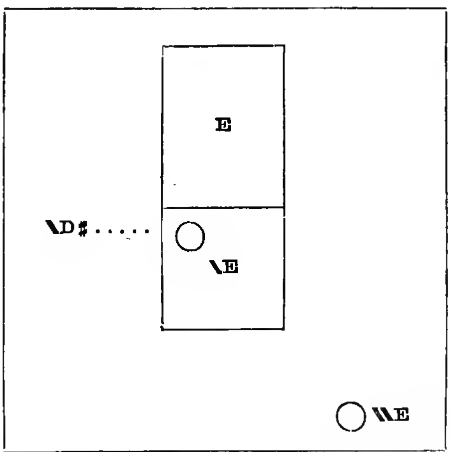
Poole's Keyboard. Arrangement of accessories.



the notes marked E, ↘E, ↘↘E, exhibit the position relation in question. Diagram VI contains the notes of the scale of C and its relative minor, with a few others. The thirds are ↘E, ↘A, ↘B; the minor thirds in the ↘A scale are C, F, G; and the major thirds occasionally required in ↘A minor are ↘C♯, ↘F♯, ↘G♯. The arrangement is completely determined by the assumption of the apotomè Pythagorica between the key-notes as a vertical step, their oblique arrangement in rows of major

VII.

Knight's-move arrangement in Colin Brown's Keyboard.



tones, and the knight's-move arrangement, as we may conveniently call it, for the derivation of the auxiliaries.

Noticing that E —↘E is a comma, and that the apotomè Pythagorica = chromatic semitone + comma, p. 10, we see that the intervals between notes of the two first series in a vertical line are alternately chromatic semitone and comma.

It is clear that this arrangement adapts itself with some facility to all music in which there is not much modulation, or in which the modulation is of a simple type.

It is however easy to give instances which will at once involve the performer in difficulties. The simple change from C major to C minor is the first instance that occurs. The /E♭ required for this change does not exist on Mr. Poole's board. He proposes however, 'if musicians decide that it is desirable to have these minor thirds,' to introduce them as additional auxiliaries. Now consider a modulation which may occur at any time,—change to C minor and modulate at once into /E♭. The performer is lost. The /E♭ is present only as an auxiliary, and cannot be used as a key-note.

Example.

Again; wherever a chord is taken with an auxiliary for its root, it will have a different form from that which it has when it has a key-note for root; though the intervals may be precisely the same.

The keyboard recently constructed by Mr. Colin Brown is of considerable interest, as being founded on exactly the same principles as Mr. Poole's. In fact, if we discard Mr. Poole's two series of harmonic sevenths, his scheme of position relations becomes absolutely identical with Mr. Brown's. This is not so odd as far as the symmetrical arrangement of the oblique rows of key-notes goes; for the writer of this treatise invented this arrangement quite independently of Mr. Poole; and as the thing is an obvious step, there is no great wonder in its having been re-invented. But Mr. Brown also applies the apotomè Pythagorica to the upward step between key-notes, as well as the knight's move arrangement * to the auxiliaries; all three arrangements being applied to the same notes, and in the same way as in Mr. Poole's instrument. The forms of the keys are a little different, but the position arrangement is absolutely identical. It appears that Mr. Brown was unacquainted with Mr. Poole's work, so that the coincidence is astonishing.

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* Diagram VII.

The objections made to Mr. Poole's keyboard apply with equal force to Mr. Brown's, except that the size of the keys in the latter is somewhat smaller, so that a more extended key-board can be provided; and it would be possible to get over the difficulty of modulation from C major to C minor and so to /E♭, by making a jump from the place on the key-board where C occurs as a key-note to the place where it occurs as an appendage to key-note /C; in which connection its minor scale and relative major are conveniently placed, just as in the case of \A minor in Diagram VI.

In modern music, however, it is often impossible to say without hesitation what the exact key relationship of a combination is.

With Poole's key-board five series of notes are required, with Colin Brown's three, to obtain the command of the combinations given by the generalised key-board with one series.

Symmetrical arrangement by Intervals.

This is the simplest principle by means of which the complex combinations of harmonious music can be analysed; its effect is to reduce all cases dealt with to a very small number of simple types.

The simplest form in which this principle is embodied is in the Duodenes of Mr. Ellis *. In these, fifths make steps in a vertical line, thirds make steps in a horizontal line. See illustrations at pp. 3, 44.

The only other form in which this principle has been applied to the writer's knowledge, is the generalised key-board which has been already described; this depends on a symmetrical arrangement by equal temperament semitones and departures therefrom. Its principles have been sufficiently explained in connection with Diagrams I and II. We will now consider the instruments in which the generalised keyboard has been applied to this class of systems.

* Proc. Royal Soc., Dec. 1874.

CHAPTER IX.

ENHARMONIC HARMONIUM.

ENHARMONIC ORGAN, POSITIVE STOP.

THE Enharmonic Harmonium exhibited at South Kensington, in the Loan Collection of Scientific Instruments 1876, was built in 1872–3. It possesses a keyboard of four and a half octaves, containing seven tiers of keys. As each tier contains twelve keys to the octave, there are altogether eighty-four keys in each octave. These are arranged in the manner indicated generally by Diagrams I and II, Diagram II giving the actual detail of a small portion of the keyboard, and Diagram I showing the general nature of the distribution in a more extended manner.

It has been mentioned that this instrument is tuned according to the division of the octave into fifty-three equal intervals, a system sensibly identical with that of perfect fifths. We will now investigate the manner in which the system is distributed on the keyboard. For this purpose we must slightly anticipate the general theory, and establish a proposition, the correctness of which we shall easily recognise.

Writing down the series of fifths about any note c, viz.—

$$d^\flat - a^\flat - e^\flat - b^\flat - f - c - g - d - a - e - b - /f^\sharp - /c^\sharp,$$

we see that there is a semitone $c - /c^\sharp$ determined by tuning seven fifths up, and there is a semitone $c - d^\flat$ determined by tuning five fifths down. These may be called seven-fifths semitones and five-fifths semitones respectively. Now scales are made up of tones and semitones, and tones are made up of two semitones each ; in other words, scales are constructed by reckoning certain numbers of semitones, whether greater or less, from a given starting point. The only general way therefore by which all possible scales whatever, in all relations to each other, can be provided for, is to have an interval for the unit of the system, which is a common divisor of the different semitones at our disposal. We have then only to find out how many and what semitones go to the octave, and this is the proposition we shall borrow from the theory.

Now with perfect fifths, the seven-fifths semitone is approximately $1\frac{7}{51}$, and the five-fifths semitone $1-\frac{5}{51}$; and the ratio of the magnitudes of these two intervals is $58:46$ or $29:23$;

and $5 \times 23 = 115$,

and $4 \times 29 = 116$;

so that the seven-fifths semitone is to the five-fifths semitone nearly as $5:4$ when made with perfect fifths.

We may therefore represent the seven-fifths semitone by five unit intervals, and the five-fifths semitone by four unit intervals without introducing any serious distortion of the fifths.

Now the theory will tell us that five seven-fifths semitones added to seven five-fifths semitones make always an exact octave. We easily see that this is true.

For five seven-fifths semitones give five E. T. semitones and the departure of thirty-five fifths upwards,

and seven five-fifths semitones give seven E. T. semitones and the departure of thirty-five fifths downwards;

leaving on the whole $7+5=12$ E. T. semitones, or an exact octave.

If we then take five units for the seven-fifths semitone,

and four for the five-fifths semitone,

five seven-fifths semitones make twenty-five units,

and seven five-fifths semitones make twenty-eight units;

and $25+28=53$;

whence fifty-three such units make an exact octave.

We proceed to construct a symmetrical arrangement, and attach to the various notes their characteristic numbers in the system of 53, according to the rule that

each seven-fifths semitone such as c—/c♯ is five units, and

each five-fifths semitone such as c—d♭ or c—c♯ is four units.

We attach as before no indication of position in the series of fifths to the different names c♯, d♭, but determine this position entirely by the notation for that purpose.

The note \\\c is taken as the first note of the series, and receives the characteristic number 1. Then c is 4, and the remaining numbers are assigned by the above process.

Diagram VIII contains a symmetrical arrangement of a portion about the middle of the keyboard, one octave in extent. It is continued both upwards and downwards on the instrument, the highest note in level being ////f, and the lowest \\\c.

It is now seen that a number of notes near the top of the keyboard are identical in pitch with other notes in the next division on the right near the bottom. This can be most simply shown by enumerating successively the characteristic numbers of the system, and tracing the succession of the notes of various names which are attached to them.

This enumeration is made in Diagram IX. On inspecting it, the following simple rule will be seen to hold, a black note meaning simply an accidental, a sharp or flat.

Rule for identifications in the system of 53.

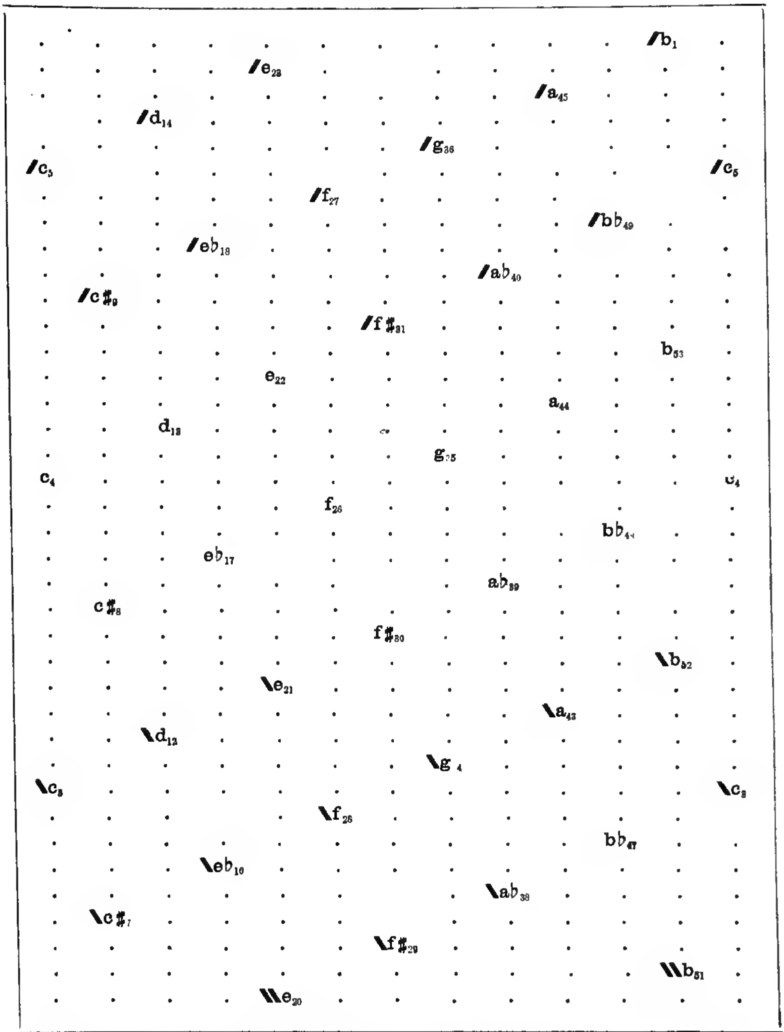
If two notes in adjoining divisions be so situated as to admit of identification (e.g. a high c and a low c♯), they will be the same if the sum of the elevation and depression marks is 4; unless the lower of the two divisions is black (accidental), then the sum of the marks of identical notes is 5.

Thus //c—\\c♯ are identical; also ///c♯—\\d.

/ is called a mark of elevation, \ a mark of depression.

The use of these identifications is to permit the infinite freedom of modulation which is characteristic of cyclical

VIII.


/b1
/e23
/a45
/d14
/g36
/c5
/c5
/f27
/bb49
/eb18
/ab40
/c#9
/f#31
b53
e22
a44
d13
g35
c4
c4
f26
bb48
eb17
ab39
c#8
f#30
\b52
\e21
\a43
\d12
\g34
\c3
\c3
\f25
bb47
\eb16
\ab38
\c#7
\f#29
\\b51
\\e20

systems. For in moving upwards on the keyboard we can, on arriving near the top, change the hands on to identical notes near the bottom, and so proceed further in the same direction, and vice versa. In perfect fifth systems, displacement upwards or downwards on the keyboard takes place

IX.

\\\c	1	/b	\\f♯	28	//f
\\c	2	//b	\f♯	29	///f
\c	3	///b	f♯	30	////f
c	4		/f♯	31	
/c	5		//f♯	32	\\\g
//c	6	\\c♯	///f♯	33	\\g
///c	7	\c♯	////f♯	34	\g
	8	c♯		35	g
	9	/c♯		36	/g
\\\d	10	//c♯	\\a♭	37	//g
\\d	11	///c♯	\a♭	38	///g
\d	12	////c♯	a♭	39	
d	13		/a♭	40	
/d	14		//a♭	41	\\\a
//d	15	\\e♭	///a♭	42	\\a
///d	16	\e♭	////a♭	43	\a
	17	e♭		44	a
	18	/e♭		45	/a
\\\e	19	//e♭	\\b♭	46	//a
\\e	20	///e♭	\b♭	47	///a
\e	21	////e♭	b♭	48	
e	22		/b♭	49	
/e	23		//b♭	50	\\\b
//e	24	\\f	///b♭	51	\\b
///e	25	\f	////b♭	52	\b
	26	f		53	b
	27	/f			

readily by modulation between related major and minor keys, not, as is commonly assumed, only by modulation round the circles of fifths. In systems of the mean-tone class, on the contrary, displacements take place only by modulations of the latter type. Consequently these last systems require a much less extended keyboard than perfect fifth systems.

The mechanism of the keyboard consists of seven tiers of levers, each tier resembling exactly the levers of an ordinary keyboard. The variations in the position of the notes are determined by the patterns of the keys which are attached to the levers. Each of these tiers communicates through a row of squares with a row of horizontal stickers*. The windchest is vertical, and the valves are arranged on it in seven horizontal rows. The valves have small tails attached, and the stickers open the valves by pressing on the tails. There is no attachment between the stickers and the valves. Thus the windchest can be lifted out by simply undoing the bolts which hold it.

The reeds are accessible by opening a large door at the back of the windchest. Each reed has a separate windchannel; and all below treble c have a regulator in the channel by which the supply of wind can be adjusted for each reed separately. In this way the usual fault of the predominance of the bass is completely obviated.

The process by which the instrument was tuned is rather complicated; an account of it is given in Proceedings of the Musical Association, 1874–5, p. 144. As far as the perfect fifths went the process was simple enough; but in order to secure the series meeting at the ends so as to give the system of 53, an elaborate system of checks was devised, the application of which was laborious. Perfect fifths are recommended by preference for ordinary purposes.

Enharmonic Organ, Positive Stop.

In the enharmonic organ with three octaves of generalised keyboard, built by the writer for the meeting of the Musical Association, May 1, 1875, the mean-tone stop of which has been already described, there is another stop occupying all the forty-eight keys per octave. It is called the 'positive' stop on the instrument. This term is applied in the general theory to systems which have fifths sharper than equal temperament; i.e. to systems which are strictly represented by such symmetrical arrangements as I or VIII; for instance perfect fifths, or the system of 53. The term 'negative' is applied, as has been

* Organ-builder's term for a rod which transmits a pressure.

mentioned, to the mean-tone class, which have fifths flatter than equal temperament, and are strictly represented by Diagram IV.

The positive stop in question was tuned on a system of the approximately perfect fifth class, the properties of which will appear in the general theory, but may be easily obtained independently.

With reference to the theory it is called the positive system of perfect thirds; sometimes it is called Helmholtz's system, as it was brought into notice by him. It differs from the system of perfect fifths only in that the third by eight fifths down is made perfect, the fifths being tempered by $\frac{1}{8}$ of the skhisma, or error of the third derived through eight perfect fifths. The skhisma being about $\frac{1}{51}$ of a semitone, the tempering of each fifth is $\frac{1}{408}$ of a semitone nearly; an interval inappreciable by ordinary means. The result of the tuning by this system instead of perfect fifths did not repay the greatly increased trouble; and in future this stop will be tuned simply by perfect fifths.

The extent and distribution of the sounds on an octave of the keyboard can be sufficiently indicated by reference to Diagram I. If we suppose the two missing notes in this scheme to be filled up (b, \\d), the /c at the top removed, and the series continued at the bottom through five more steps down to \\\c, we shall have a representation of the distribution of the sounds on one octave of this stop.

The result of this stop on this organ is not so satisfactory as that of the mean-tone stop, and not nearly so satisfactory, in the writer's judgment, as the result on the 53 harmonium. It should be mentioned that there is a difference of opinion on this subject owing to the dislike which some persons entertain to the somewhat sharp quality of the harmonium reeds. The large majority of persons however prefer the harmonium. The reason is certain: the pipes of the organ are metal stopped diapasons, and they possess of course the smooth quality characteristic of such pipes. With this particular quality of tone, and indeed with diapason tone in general, little seems to be gained by the degree of additional accuracy which the perfect-fifth systems possess over the mean-tone class. It is not possible with the particular stop in question to illustrate in a striking manner the difference between chords in and out of tune; the quality is too smooth to be very sensitive to tuning, and the general effect is somewhat monotonous.

General Thompson's organ, which is now* at the Loan Exhibition, South Kensington, is open to the same remarks. The quality of tone is somewhat firmer than that of the stopped pipes of the writer's organ; but after becoming acquainted practically with General Thompson's organ, as well as hearing it performed upon a good deal by persons who for the most part did not understand it, and merely flourished about at random on the keyboard, the writer is prepared to maintain that the gain in the purity of the chords is hardly enough to make it worth while to face the enormous cost and demand for space which must be inseparable conditions of the application of perfect-fifth systems to the organ on any very considerable scale. The mean-tone system seems more applicable to this purpose, while the perfect-fifth systems have special applicability to the harmonium, and also, in all probability, have a wide field before them in connection with the orchestra.

The small enharmonic organ which contains the two abovementioned stops is designed on a principle which is general, and susceptible of extension to instruments of greater size. The keyboard is arranged in four tiers; the tails of each tier fit without attachment under the fronts of a row of squares, the other corners of which hang down, and pull directly on the principal trackers. In consequence of this arrangement the entire keyboard can be lifted out and replaced in a couple of minutes. The principal trackers run from the lower ends of the squares, parallel with the keys, forming four tiers corresponding to the four tiers of keys, and engaging at the other ends in four tiers of rollers; each roller is parallel to the width of the keyboard, or at right angles to the length of

* 1876.

the key and tracker, and the length of the roller is a little greater than the width of the keyboard. The windchest is above the roller board; it is about 2 ft. 6 in. wide, and 6 ft. from back to front; the keys are in the narrow front. The pull-downs are arranged in two rows along the long sides of the windchest, and come down to the ends of the rollers on either side alternately. Thus any valve or pull-down can be got at at once. The stop sliders run parallel to the keys and trackers, and to the greatest length of the windchest, from front to back, so that the action is what is called 'direct.'

The stoppers of the metal pipes employed are a novelty. They are put together in the first instance like square wooden pipes, with a square channel in the middle of each; they are then turned in the lathe to fit the pipes they are to stop. A square block is fitted tightly into the interior channel, and a screw fitted into the block. A headpiece through which the screw passes completes the stopper. The object is to get a fine adjustment for tuning. When the pipe is nearly right it can be very minutely adjusted by means of the screw.

CHAPTER X.

GENERAL THEORY OF THE DIVISION OF THE OCTAVE*.

Definitions.

Regular Systems are such that all their notes can be arranged in a continuous series of equal fifths.

Regular Cyclical Systems are not only regular, but return into the same pitch after a certain number of fifths. Every such system divides the octave into a certain number of equal intervals.

Error is deviation from a perfect concord.

Departure is deviation from an equal-temperament interval.

Equal temperament (E. T.) is the division of the octave into twelve equal intervals.

Intervals are expressed in terms of equal-temperament semitones; so that the octave is written as 12, and the E. T. semitone as 1.

Intervals taken upwards are called positive, taken downwards, negative.

Fifths are called positive if they have positive departures, i.e. if they are greater than E. T. fifths; they are called negative if they have negative departures, i.e. if they are less than E. T. fifths. Perfect fifths are more than seven semitones; they are therefore positive.

Systems are said to be positive or negative according as their fifths are positive or negative. (See Diagram VIII or I for positive systems, IV for negative systems.)

Regular cyclical systems are said to be of the r^{th} order,

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* This Chapter may be omitted if it is desired to confine the attention to the practical part of the subject.

positive or negative, when twelve of the approximate fifths of the system exceed or fall short of seven octaves by r units of the system.

Thus if thirty-one units of the system of 53 be the fifth,

then $12 \times 31 = 372$,

$7 \times 53 = 371$,

the twelve fifths exceed seven octaves by one unit, and the system is said to be of the first order positive.

We shall see later that in the system of 118 the twelve fifths exceed seven octaves by two units, and the system is said to be positive of the second order.

In the system of 31, twelve fifths fall short of the octave by one unit, and the system is said to be negative of the first order.

In the system of 50, twelve fifths fall short of seven octaves by two units, and the system is said to be negative of the second order.

COR. Hence the departure of twelve fifths is r units of the system, having regard to sign.

Regular Systems.

THEOREM I. In any regular system five seven-fifths semitones and seven five-fifths semitones make up an exact octave.

For the departures from E. T. of the five seven-fifths semitones are due to thirty-five fifths up,

and the departures of the seven five-fifths semitones are due to thirty-five fifths down,

leaving twelve E. T. semitones, which form an exact octave.

(This has been proved already in connection with the system of 53, but it is necessary to repeat it here, as it forms the foundation of the theory.)

THEOREM II. In any regular system the difference between the seven-fifths semitone and the five-fifths semitone is the departure of twelve fifths, having regard to sign.

For if we subtract the five-fifths semitone from the seven-fifths semitone, the E. T. semitones cancel each other;

and the departure of the seven-fifths semitone up is due to seven fifths up,

and that of the five-fifths semitone down is due to five more fifths up,

making the departure of twelve fifths:

and it is positive if the fifths are positive, and negative if the fifths are negative.

Regular Cyclical Systems.

THEOREM III. In a regular cyclical system of order $\pm r$, the difference between the seven-fifths semitone and five-fifths semitone is $\pm r$ units of the system.

This proposition follows from Th. II, and the Cor. to the definition of r^{th} order.

COR. This proposition, taken with Th. I, enables us to determine the numbers of divisions in the octave in systems of any order, by introducing the consideration that each semitone must consist of an integral number of units. The principal known systems are here enumerated:—

PRIMARY (1ST ORDER) POSITIVE.

Seven-fifths semitone $= x$ units.	Five-fifths semitone $= y$ units.	Number of units in octave (Th. I). $5x + 7y = n$.
2	1	17
3	2	29
4	3	41
5	4	53
6	5	65
SECONDARY (2ND ORDER) POSITIVE.		
11	9	118
PRIMARY NEGATIVE.		
1	2	19
2	3	31
SECONDARY NEGATIVE.		
3	5	50

The mode of formation in other cases is obvious.

THEOREM IV. In any regular cyclical system, if the octave be divided into n equal intervals, and r be the order of the system, the departure of each fifth of the system is $\frac{r}{n}$ E. T. semitones.

Let the departure of each fifth of the system be δ. Then the departure of twelve fifths $=12\delta=r$ units by definition or its corollary; and the unit $=\frac{12}{n}$ E. T. semitones (since the octave, which is twelve semitones, is divided into n equal parts). Hence—

$$12\,\delta=r\,.\frac{12}{n}, \quad \text{or} \quad \delta=\frac{r}{n}\cdot$$

We have seen that the departure of each fifth of the system of 53 is $\frac{1}{53}$ of an E. T. semitone; this is a particular case of the above theorem.

As a consequence of this theorem we can shew that the system of 31 is nearly the same as the mean-tone system. For the departure of the fifth of the mean-tone system is $\frac{1}{29.2}$ downward (p. 35), or $-\frac{1}{29.2}$, and by the above Theorem the departure of the fifth of the system of 31, for which $r=-1$, is $-\frac{1}{31}$: and the two differ by an amount which is insensible in practice.

THEOREM V. If in a system of the r^{th} order, the octave be divided into n equal intervals, $r+7n$ is a multiple of 12, and $\frac{r+7n}{12}$ is the number of units in the fifth of the system.

Let ϕ be the number of units in the fifth.

Then $\phi\,.\,\frac{12}{n}$ is the fifth, $=7+\delta$, if δ be the departure of one fifth; and $\delta=\frac{r}{n}$ by Th. IV.

Hence $$\phi\,.\frac{12}{n}=7+\frac{r}{n}, \quad \text{or} \quad \phi=\frac{7n+r}{12},$$

and ϕ is an integer by hypothesis; whence the proposition.

From this proposition we can deduce corresponding values of n and r. Casting out multiples of 12, where necessary, from n and r, we have the following relations between the remainders:—

n	1	2	3	4	5	6	7	8	9	10	11
r	5	10	3	8	1	6	11	4	9	2	7
	—7	—2	—9	—4	—11	—6	—1	—8	—3	—10	—5.

Example. It is required to find the order of the system in which the octave is divided into 301 equal intervals. 300 is a multiple of 12; remainder 1 gives order 5, or —7. 301 is a system of some interest regarded as a positive system of order 5, in consequence of its having tolerably good fifths and thirds, while its intervals are expressed by the first three places of the logarithms of the vibration ratios, .3010 being the first four places of log 2. Mr. Ellis has made use of this system (Proceedings of Royal Society 1874); and Mr. Pole read a paper about it to the Musical Association 1875–6.

THEOREM VI. If a system divide the octave into n equal intervals, the total departure of all the n fifths of the system $= r$ E. T. semitones, where r is the order of the system.

For if δ be the departure of one fifth, then, by Th. IV,

$$\delta = \frac{r}{n}; \quad \text{whence } n\delta = r,$$

or the departure of n fifths $= r$ semitones.

This theorem gives rise to a curious mode of deriving the different systems.

Suppose the notes of an E. T. series arranged on a horizontal line in the order of a succession of fifths, and proceeding onwards indefinitely thus :—

c g d a e b f♯ c♯ g♯ d♯ a♯ f c g . . .

and so on.

Let a regular system of fifths start from c. If they are positive, then at each step the pitch rises further from E. T. It can only return to c by sharpening an E. T. note.

Suppose that b is sharpened one E. T. semitone, so as to become c; then the return may be effected—

at the	first b	in	5 fifths
„	second b	„	17 fifths
„	third	„	29 fifths
„	fourth	„	41 fifths
„	fifth	„	53 fifths,

and so on. Thus we obtain the primary positive systems. Secondary positive systems may be obtained by sharpening b♭ by two semitones, and so on.

If the fifths are negative, the return may be effected by depressing c♯ a semitone in 7, 19, 31 . . . fifths; we thus obtain the primary negative systems; or by depressing d two semitones, by which we get the secondary negative systems, and so on.

THEOREM VII. If n be the number of divisions in the octave in a system of the r^{th} order, then $n+7r$ will be divisible by 12, and $\frac{n+7r}{12}$ will be the number of units in the seven-fifths semitone of the system.

For by the order condition (Th. V) $7n+r$ is a multiple of 12; whence $7(7n+r)=49n+7r$ is a multiple of 12; whence, casting out $48n$,

$$n+7r \text{ is a multiple of } 12.$$

Let x be the number of units in the seven-fifths semitone, then

$$x.\frac{12}{n}=1+7\delta=1+7\frac{r}{n},$$

whence
$$x=\frac{n+7r}{12},$$

and the proposition is proved.

THEOREM VIII. Negative systems form their major thirds by four fifths up.

For the departure of the perfect third is $-.13686$ or $-\frac{1}{7.3}$ approximately; that is, it falls short of the E. T. third by this fraction of an E. T. semitone. But in negative systems the fifth is of the form $7-\delta$; and four fifths less two octaves give $4(7-\delta)-24=4-4\delta$, a third with negative departure, which can be determined so as to approximate to the perfect third.

COR. I. The mean-tone system may be derived from this result by putting

$$-4\delta=-.13686$$
$$-\delta=-.034215$$

COR. II. The departure of a third of a negative cyclical system n of order $-r$ is $-4\frac{r}{n}$.

THEOREM IX. Positive systems form approximately perfect thirds by eight fifths down.

The departure of the perfect third is $-.13686$.

But in positive systems the fifth is of the form $7+\delta$; and five octaves up and eight fifths down give $60-8(7+\delta)=4-8\delta$, a third with negative departure, which can be determined so as to approximate to the perfect third.

COR. I. The positive system of perfect thirds, or Helmholtz's system, can be derived from this result by putting

$$-8\delta = -.13686$$
$$\delta = .0171075.$$

COR. II. The departure of a third of a positive cyclical system n of order r is $-8\frac{r}{n}$.

THEOREM X. Helmholtz's Theorem. The third thus formed with perfect fifths has an error nearly equal in amount to the error of the E. T. fifth.

For $-8\times.01955= -.15640$

$$\begin{array}{r} -.13686 \\ \hline -.01954 \end{array}$$ which is nearly $= .-01955$.

The quantity $.01954$ is called the skhisma.

THEOREM XI. In positive systems an approximate harmonic seventh can be obtained by fourteen fifths down.

The departure of the harmonic seventh is $-.31174$; and fourteen fifths down and nine octaves up give,

$$108-14(7+\delta)=10-14\delta,$$

a minor seventh with negative departure.

THEOREM XII. In negative systems an approximate harmonic seventh can be obtained by ten fifths up. For five octaves down and ten fifths up give

$$10(7-\delta)-60=10-10\delta,$$

a minor seventh with negative departure.

Concords of Regular and Regular Cyclical Systems.

These considerations permit us to calculate the departures and errors of concords in the various regular and regular cyclical systems. There is, however, one other quantity which may be also conveniently taken into consideration in all cases, viz. the departure of twelve fifths of the system. We will call this Δ, putting $\Delta = 12\delta$.

We have then the following table of the characteristic quantities for the more important systems hitherto known.

The value of the ordinary comma $\left(\frac{81}{80}\right)$ is ·21506. It is comparable with the values of Δ, and if introduced in its place in the table would give rise to a regular non-cyclical system, lying between the system of 53 and the positive system of perfect thirds, the condition of which would be that the departure of twelve fifths = a comma.

Name, or n.	Order, r.	$\Delta = 12\delta$, or $12\frac{r}{n}$.	Error of fifth, $\delta - {\cdot}01955$	Error of third, ${\cdot}13686 - 8\delta$.	Error of harmonic seventh, ${\cdot}31174 - 14\delta$.
17	1	·70588	·03927	−·33373	−·51178
29	1	·41379	·01493	−·27586	−·17101
41	1	·29268	·00484	−·19512	−·02970
Perfect fifths.	...	·23460	...	−·01954	·03804
53	1	·22642	−·00068	−·01409	·04758
Positive perfect thirds.	...	·20529	−·00244	...	·07223
118	2	·20339	−·00260	·00127	·07445
65	1	·18462	−·00417	·01378	·09635
($\delta = \frac{r}{n}$ is here negative.)				.13686 + 4δ	.31174 + 10δ
43	−1	−·29707	−·04431	·03784	·06418
31	−1	−·38710	−·05181	·00783	−·01084
Mean Tone. Negative perfect thirds.	...	−·41058	−·05376	...	−·03041
50	−2	−·48000	−·05955	−·02314	−·08826
19	−1	−·63158	−·07218	−·05367	−·21458

THEOREM XIII. If a symmetrical arrangement like Diagram I or VIII be constructed, the dots being all considered as notes, and the vertical distance between two dots represent $\frac{1}{51}$ of an E. T. semitone, the whole system will constitute a division of the octave into 612 equal intervals*, and it will possess both fifths and thirds correct to a high order of approximation.

For since the octave is twelve semitones, and the semitone fifty-one units, the octave is 612 units.

Again the system may be regarded as made up of fifty-one different sets of E. T. notes, each represented by the dots of a horizontal line. The fifth, the upper note of which is one step above the lower, will be $7\frac{1}{51}$, and the perfect fifth is $7\frac{1}{51.151}$, a very small difference.

The third, which has its upper note seven steps below its lower note, is $4-\frac{7}{51}$ or $4-\frac{1}{7.286}$, and the perfect third is $4-\frac{1}{7.3064}$, also a very small difference.

A symmetrical arrangement with all the positions filled in in this manner may be called a complete symmetrical arrangement. It might be constructed with concertina keys.

* The importance of this system was pointed out by Captain J. Herschel, F.R.S.

CHAPTER XI.

MUSICAL EMPLOYMENT OF POSITIVE SYSTEMS HAVING PERFECT OR APPROXIMATELY PERFECT FIFTHS.

THE following example is repeated here from p. 12, as containing examples of the principal forms of chords.

The first chord is the major triad of c.

The second chord is the triad of the dominant g, with ╲f, the approximate harmonic seventh, as dominant seventh.

The last two crotchets of the first bar are the chords of c major and minor.

The chord at the beginning of the second bar is the augmented sixth, rendered peculiarly smooth in its effect by employment of the approximate harmonic seventh for the interval (╱a♭—f♯).

We proceed to notice practical points affecting the employment of the principal intervals.

Second of the Key.—In any positive system the second of the key may be derived in two ways: first, as a fifth to the dominant, in which case the derivation is by two fifths up from the key-note; and, secondly, as a major sixth to the subdominant, in which case the derivation is by ten fifths down from the key-note. Thus, the first second to c is d; the other ╲d. On account of the importance of this double form of second, we will consider the derivation of these two forms by means of the ordinary ratios, in the case, namely, in which perfect intervals are employed.

First, two fifths up and an octave down give

$$\left(\frac{3}{2}\right)^2 \times \frac{1}{2} = \frac{9}{8},$$

when the fifths are perfect.

Secondly, one fifth down gives the subdominant (c—f), and a sixth up gives the depressed second (\d), or

$$\frac{2}{3} \times \frac{5}{3} = \frac{10}{9},$$

which is the ratio of \d to the keynote, when the fifths and thirds are perfect.

The ratio of d : \d is then

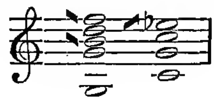
$$\frac{8}{9} \div \frac{10}{9} = \frac{81}{80},$$

which is an ordinary comma.

We must remember that our systems only give approximations to this result, but the best of these approximations are very close.

In the harmonium, with the system of 53—which may be regarded for practical purposes as having perfect fifths, and very nearly perfect thirds—the exchange of d for \d in the chord f—\a—\d, or even in the bare sixth, f—\d, produces an effect of dissonance intolerable to most ears.

Minor Third.—The minor third is not an interval which is very strictly defined by beats. In chords formed of successions of minor thirds, almost any form of the interval may be employed; and as matter of fact the minor third which comes below the harmonic seventh in the series of harmonics (7 : 6), is one of the smoothest forms of this interval. c—\e♭ is an approximation to such a chord, where the \e♭ is derived by fifteen fifths down. But in minor common chords the condition is that the major third or sixth involved shall be approximately perfect; and this gives the triad c—/e♭—g where the /e♭ is derived by nine fifths up. The intermediate form, e♭, gives a minor third not quite so smooth as either of the other two; but it is capable of being usefully employed in such combinations as the diminished seventh, and it is preferred by many listeners, as deviating less from the ordinary equal temperament note, from which it has only the departure due to three fifths down. The interval between the harmonic seventh on the dominant and the minor third of the elevated form on the keynote, is the smallest value of the whole tone which occurs, the departure from E. T. of such a tone being due to twenty-two fifths or about two commas; and although two chords, involving these notes in succession, may each be perfectly harmonious, the sequence is generally offensive to ears accustomed to the equal temperament.

Example. 

Custom makes such passages sound effective, especially when the succession is slow enough to enable the ear to realise the fineness of the chords.

Major Third.—This interval has been already discussed; the note taken is that formed by eight fifths down.

Fourths and Fifths need no remark.

Depressed Form of the Dominant.—When the dominant is used in such a combination as the following:—

it must be formed by eleven fifths down from the key-note, unless we regard the key-note as changed for the moment, in which case, by elevating the subdominant, we may retain the fifth in its normal position. The most judicious course depends on whether the fifth is suspended or not. Thus, if the fifth is suspended, we may write:—

For if the subdominant be f, its third must be \a, and its sixth must be \d; g then makes a fourth with \d, which is unbearable to the ear; the fourth must be made correct, and the ways of doing so are shown above. The difficulty may be otherwise got over by writing the passage


&c.

Minor Sixth.—This interval is pretty sharply defined. The usual form is /a♭, which is got by eight fifths up; the key-note forms an approximately perfect third with this note by inversion.

Major Sixth (c—\a).—This interval is, as a matter of fact, more sharply defined than one would expect. This interval must be kept strictly to its best value. The \a is got by nine fifths down.

In chords formed of a succession of minor thirds, major sixths frequently occur. Care must be taken to dispose them so as to make this interval correct. If a deviation is necessary, it is better, if possible, to extend the interval by an octave; the resulting major thirteenth (3 : 10) is not very sensitive.

Minor Seventh.—There are three forms of the minor seventh. To fundamental c these are are /b♭, b♭, and \b♭:—

/b♭; ten fifths up; the minor third to the dominant.
b♭; two fifths down; the fourth to the subdominant.
\b♭; fourteen fifths down; approximation to the harmonic or natural seventh.

Rule.—The natural or harmonic seventh on the dominant must not be suspended, so as to form a fourth with the key-note.

Major Seventh.—There is only one form of major seventh which can be used in harmony, viz. \b; this note is got by seven fifths down; it forms a major third to the dominant. In unaccompanied melody the form b produces a good effect. This is got by five fifths up with perfect fifths. It forms a dissonant or Pythagorean third to the dominant. The resulting semitone is less than the E.T. semitone by nearly $\frac{1}{10}$ of a semitone.

An example of music written for positive systems is appended, p. 77.

The principal points in the harmony of these systems which have struck the writer occur in the example. It is to be specially noticed how certain forms of suspension have to be avoided—partly because they produce dissonances, partly because they occasion large displacements up and down the keyboard. The result of the writer's practical experience is, distinctly, that there are many passages in ordinary music which cannot be adapted with good effect to positive systems; and that the rich and sweet masses of tone which characterise these systems, with the delicate shades of intonation which they have at command, offer to the composer a material hitherto unworked. The character of music adapted for these systems is that of simple harmony and slow movement; it is a waste of resources to attempt rapid music, for the excellence of the harmonies cannot be heard. The mean-tone system is more suitable for such purposes.

Some examples of the unsuitability of the positive systems for ordinary music may be first instanced:—

(1) The opening bars of the first prelude of Bach's 48. The second bar involves the depressed second (\d), and in the third bar this changes to d; the melodic effect is extremely disagreeable on the harmonium. It does not strike the ear much with the stopped pipes of the little organ.

(2)

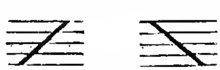
The two g's, to which attention is here called by asterisks, illustrate a difficulty of constant occurrence in the adaptation of ordinary music to these systems. The g is here required to make a fourth to the depressed second of the key (\d), and also a fifth to the keynote. But the first condition requires the note ↘g, the second g, and it is impossible to avoid the error of a comma somewhere. It may be said that the first g is only a passing note; but with the keen tones of the harmonium such dissonances strike through everything, even on the least emphasised passing notes. Although the second g seems to the writer to be legitimate, it would be intolerable on the harmonium. The smoother tones of the organ render such effects less prominent.

(3) The third phrase of a well-known chant:—

To keep in the key of f, the g should fall to ↘g at the second chord; but this direct descent on the suspended note would sound bad—consequently, the whole pitch is raised a comma at this point by the suspension; and the chant concludes in the key of ↗f, as it is not possible anywhere to descend again with good effect. This would be inadmissible in practice, as the pitch would rise a comma at each repetition. The resources of the system of 53 admit of the performance of repetitions in this manner, but the case is one in which the employment of this effect would be unsuitable.

On the organ it might be possible to take the last chord written above ↘g—↘d—↘g—b♭, which would get rid of the difficulty. On the harmonium, however, this drop from the minor chord of g to that of ↘g is inadmissible.

In the example of music written for the positive systems, it is to be noted that the notation-marks are used as signatures, exactly as flats and sharps are in ordinary music. The sign adopted for neutralising them is a small circle (○), which is analogous to the ordinary natural. If the general pitch had to be raised or depressed by a comma, the elevation or depression mark would be written large over the beginning of the staff:—



Several points in the harmony are regarded as experimental. For instance, in the inversion of the dominant seventh with the seventh in the bass, the employment of the depressed (harmonic) seventh has on the harmonium an odd effect; although, when the chord is dwelt on, it is heard to be decidedly smoother than with the ordinary seventh. The effect appears less strange on the organ. On this and other points the judgment of cultivated ears must be sought, after thorough acquaintance with the systems.

The following points may be noticed in the example at p. 77.

At the beginning of the seventh bar it would be natural, in ordinary music, to suspend the a, from the preceding chord, thus:—

As however the first a is \a, and we are modulating into g, whose dominant is d, the suspension is inadmissible, as it would lead to the false fifth d—\a.

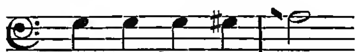
In bar 14 the ordinary seventh \d to dominant \e is employed in the bass instead of the harmonic seventh \\d, so as to avoid the small tone \\d—c. The latter has a bad effect in the minor key, as before noticed, and this is specially marked in the bass.

Bar 19.—The use of the tonic as first note in the bass is prevented by the presence of the harmonic seventh on the dominant, p. 42.

Bar 24.—This singular change is pleasing in its effect when judiciously used, but it is advisable to separate the two forms of the chord by a rest.

Bar 30.—The smoothness of the approximate harmonic seventh is here applied to the sharp sixth. This effect is the most splendid which the new systems afford; nothing like it is attainable on ordinary instruments.

Bars 34 *and* 35.—Here the natural course would be to make the bass:—

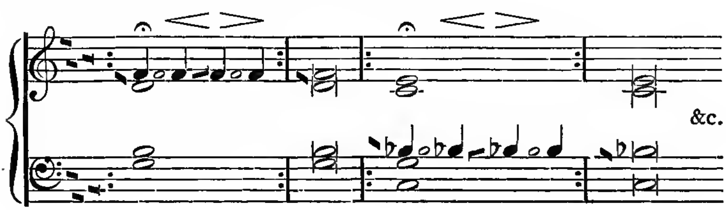


the harmony remaining the same. We have however arrived at our d as the fifth to g, and it is not possible to suspend it unless we raise the \a to a. It has not a good effect where a passage is repeated as here, if the repetition is in a slightly different pitch. The suspension is therefore avoided.

Bar 37.—This is a very charming effect. The transient modulation to dominant d gives the depressed key-note, \c, as harmonic seventh.

Comma Scale.

The following is an example of a novel effect which is attainable in positive systems. If the chord of the harmonic or natural seventh be sustained, this seventh may be made to rise and fall again through two or more single commas. The effect to unaccustomed ears is disagreeable at first; but the writer has become so familiar with these small intervals, that he hears them as separate notes without the sensation they commonly produce of being one and the same note put out of tune. There can be no doubt that the reception of such intervals is a question of education, just as the reception of semitones was, in the early history of music, a step in advance from the early five-note scales. The following passage, as executed on the enharmonic harmonium, which admits of a swell of the tone, has a dramatic effect:—



Series of Major Thirds.

The chord to which attention is called consists of two perfect thirds and the octave. The third \g♯—c has a departure due to sixteen fifths up, and an error from the perfect third of about two commas. It may be called the 'superdissonant' third, by analogy from the dissonant or Pythagorean third, which has an error of one comma. We have the choice, if we prefer it, of arranging the chord with two dissonant thirds, thus :—

c—\e—g♯—c.

The two last thirds are ordinary dissonant thirds; the writer prefers the first arrangement. It is a matter of taste.

EXAMPLE FOR SYSTEMS OF APPROXIMATELY PERFECT FIFTHS, WITH A COMPASS OF THREE OCTAVES.

H = *Harmonic or Natural Seventh, or inversion thereof.*


3
H
14
H
H
19
H
24
H
30
H


H
H
34
H
H
H
37
H
H

APPENDIX.

ON THE THEORY OF THE CALCULATION OF INTERVALS.

WHEN we consider the interval between two notes, with reference either to the relation between their vibration numbers, or between the lengths of a string which will sound them, we employ the ratio between the numbers in question, that is, we divide the one number by the other.

Thus, if we take lengths of a stretched string which are as 1 : 2, they produce notes an octave apart. If we take lengths as 2 : 3, they produce a fifth; and if we take lengths as 4 : 5, they produce a third. The case is the same with the vibration numbers.

If we desire to estimate the interval formed by the sum of two others, retaining for clearness the conception of the lengths of string, we see that $\frac{1}{2}$ the string will give us the octave, and $\frac{2}{3}$ of that $\frac{1}{2}$, or $\frac{2}{3} \times \frac{1}{2}$ of the whole length, gives us the sum of octave and fifth: that is to say, in order to find the string fraction for the sum of two intervals, we have to multiply together their separate string fractions.

This refers the principle of multiplication of ratios directly to our experimental knowledge of the properties of fractional lengths of a musical string.

In the same way the same rule for vibration ratios may be referred directly to our experimental knowledge of the properties of vibration numbers; or, more simply, the latter may be deduced from the fact, known from the laws of mechanics, that the vibration numbers are inversely as the string lengths of portions of the same string.

When we perform computations in this manner by multiplication and division of ratios, the numbers are apt to become high, the computations troublesome or impossible (e. g. the division of the octave into 53 equal intervals), and the appreciation of the magnitude of the intervals in question difficult. It is, in particular, difficult to interpret the results of a fractional computation in terms of such intervals as are in practical use, e. g. equal temperament semitones.

In order to overcome these difficulties, methods are adopted which are explained at length, in principle and practice, in the remarks which follow.

If we take the ratio corresponding to any interval (e. g. the ratio 2, corresponding to an octave), multiply it by itself over and over again, and then set down the resulting products, with the number of times the multiplication has been performed, in two columns, we form two corresponding series, the one in geometrical, the other in arithmetical progression, thus:—

Geometrical Progression of ratios.	Arithmetical Progression of number of ratios.
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8

and so on.

Again, let x be the ratio corresponding to an equal temperament semitone, that is to say, let x^{12}, or x multiplied 12 times into itself, be equal to 2. Then, forming a series like the above, we have :—

Geometrical Progression of ratios.	Arithmetical Progression of E. T. semitones.
1	0
x	1
x^2	2
x^3	3
x^4	4
x^5	5
x^6	6
x^7	7
x^8	8
x^9	9
x^{10}	10
x^{11}	11
$2=x^{12}$	12

and so on.

Now, whenever two sets of numbers form corresponding terms in a pair of series of this kind, the numbers in the arithmetical progression are called logarithms of the corresponding numbers in the geometrical progression; and the number in the geometrical progression which corresponds to logarithm 1 is called the base of the system of logarithms.

Equal temperament semitones may therefore be regarded as logarithms of vibration ratios to base x, where $x^{12}=2$.

Common logarithms, such as are found in the ordinary tables, are to base 10.

We can find x independently of ordinary logarithms by using the ordinary processes of square and cube root.

For since $x^{12}=2$

$$x = \sqrt[12]{2}$$

$$= \sqrt[2]{\sqrt[2]{\sqrt[3]{2}}}$$

The arithmetician may therefore find x for himself by twice extracting the square root of 2, and then the cube root of the result; the numbers x^2, x^3, x^4, and so on, are easily obtained when this operation has been performed.

In practice we derive these numbers more shortly by making use of the labours of those who constructed our tables of common logarithms, in the manner explained in the text.

Passing for a moment from the subject of the construction of such tables as the above, let us see what use can be made of them. We can only speak of intervals made up of E. T. semitones with reference to the above illustration, but this will be sufficient for the present purpose.

Suppose we have got the numbers x^2 and x^5, the first being the ratio of two semitones, and the second the ratio of five semitones: first, to find the sum of these two intervals. If we had only the numbers, we should have to multiply them together, and interpret the result as best we could; but having the table, we have only to add together the numbers of semitones, or the logarithms, and we not only learn how many semitones the resulting interval consists of, but can find opposite that number (7), in the table, the number which x^2 and x^5 would give if multiplied out. Instead of multiplying the ratios we add the logarithms. Similarly, if we wish to divide one ratio by another, we subtract the logarithms.

Again, we can use these numbers for dividing an interval into any number of equal parts.

Thus if we want to divide the octave into two equal parts, with the ratios we should have to take the square root of 2. We perform the same process here by simply dividing 12 by 2, and noting the ratio opposite the result in the table.

Similarly the multiplication of a ratio by itself any number of times is reduced to multiplying the number of equal semitones by the number of times the multiplication is desired to be performed. Thus the E. T. fifth is seven semitones. If we want to find the value of twelve such fifths we have only to multiply 7 by 12, which gives 84, or seven octaves.

By means of a system of logarithms then, we reduce—

multiplication of ratios	to addition,
division	to subtraction,
extraction of a root .	to division, and
raising to a power . .	to multiplication.

Practically, for musical purposes, we do not construct a table of this kind. The ratios we have to deal with can be reduced to two or three, which we turn into E. T. semitones most simply by employing ordinary logarithms: and when we once know these equivalents we can form the rest by their means. Thus if we know the values of the fifth and third in E. T. semitones, we can form any of the intervals ordinarily discussed in connection with the diatonic scale, by addition, subtraction, and multiplication and division by low numbers.

I shall however proceed, for the satisfaction of those who are not acquainted with logarithms, to develope methods, by which an arithmetician may perform the computations for himself.

The first method of proceeding further is very simple in principle; and it is interesting as being, in principle, very like the method which was actually used by the first constructors of logarithms. It is however so laborious to carry out that we will dispense with the execution of the calculations.

It consists simply of obtaining equivalent ratios for fractions of equal temperament semitones by continual extractions of square roots. Proceeding in this manner, we should obtain the following equivalents, taking square roots on the left, and dividing by 2 on the right:—

Ratio.	Decimal of E. T. semitone.
$\sqrt[2]{x}$	.5
$\sqrt[4]{x}$	·25
$\sqrt[8]{x}$	·125
$\sqrt[16]{x}$	·0625
$\sqrt[32]{x}$	·03125
$\sqrt[64]{x}$	·015625
$\sqrt[128]{x}$	·0078
$\sqrt[256]{x}$	·00390625
$\sqrt[512]{x}$	·001953125

and so on.

For the construction of a table of practical utility it would be necessary to proceed further.

Now suppose we want to construct the ratio answering to 1·1 (one and a tenth) E. T. semitones, we must make this up out of the terms we have found.

Thus	1	is represented by			x
	·0625	,,		,,	$\sqrt[16]{x}$
	·03125	,,		,,	$\sqrt[32]{x}$
	·00390625	,,		,,	$\sqrt[256]{x}$
	·001953125	,,		,,	$\sqrt[512]{x}$
	1·099609375				

So that a value within $\frac{1}{1000}$ of a semitone of the required value 1·1 is equivalent to the ratio

$$x \times \sqrt[16]{x} \times \sqrt[32]{x} \times \sqrt[256]{x} \times \sqrt[512]{x}.$$

For the practical construction of tables these approximations require to be carried further.

This process not being suitable for actual use, I proceed to explain a method analogous to that which would be now employed for the independent calculation of logarithms.

In treatises on the construction of logarithms, such as occur in the ordinary books on trigonometry, it is proved that, in logarithms to any base, if

x be the number whose logarithm is sought,

C a number which is always the same for the same system of logarithms ;—

then,

$$\log x = C\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\ldots\right\}.$$

Where the successive terms become smaller and smaller, and after a certain point cease to influence the computation.

Suppose we have a ratio $\frac{a}{b}$ to deal with ; then if we put

$$x = \frac{a}{b}$$

the above becomes ;

$$\text{(A)}\ldots\ldots\log\frac{a}{b} = C\left\{\frac{a-b}{a+b} + \frac{1}{3}\left(\frac{a-b}{a+b}\right)^3 + \frac{1}{5}\left(\frac{a-b}{a+b}\right)^5 + \ldots\ldots\right\}.$$

In order to use this formula we have only to determine C; and this can be always done by making equation (A) satisfy the law of the given system.

Thus the law of the system of E. T. semitones is, that the logarithm of 2 must be 12.

Putting $a=2$, $b=1$,

$$\frac{a-b}{a+b} = \frac{1}{3} = \cdot 333\ 3333$$

$$\left(\frac{a-b}{a+b}\right)^3 = \frac{1}{3\times 9} = \cdot 037\ 0370$$

$$\left(\frac{a-b}{a+b}\right)^5 = \frac{1}{3\times 9^2} = \cdot 004\ 1152$$

$$\left(\frac{a-b}{a+b}\right)^7 = \frac{1}{3\times 9^3} = \cdot 000\ 4572$$

$$\left(\frac{a-b}{a+b}\right)^9 = \frac{1}{3\times 9^4} = \cdot 000\ 0508$$

$$\left(\frac{a-b}{a+b}\right)^{11} = \frac{1}{3\times 9^5} = \cdot 000\ 0056$$

Whence

$$\left(\frac{a-b}{a+b}\right) = \cdot 333\ 3333$$

$$\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3 = \cdot 012\ 3457$$

$$\frac{1}{5}\left(\frac{a-b}{a+b}\right)^5 = \cdot 000\ 8230$$

$$\frac{1}{7}\left(\frac{a-b}{a+b}\right)^7 = \cdot 000\ 0653$$

$$\frac{1}{9}\left(\frac{a-b}{a+b}\right)^9 = \cdot 000\ 0056$$

$$\frac{1}{11}\left(\frac{a-b}{a+b}\right)^{11} = \cdot 000\ 0005$$

$$\overline{\cdot 346\ 5734}$$

is the value of the series for the interval of the octave; putting the logarithm $=12$, we determine C by means of the equation (A).

$$12 = C \times \cdot 346\ 5734$$

whence

$$C = \frac{12}{\cdot 346\ 5734}$$

$$= 34\cdot 62469$$

which is therefore the value of the constant C in equation (A), for the computation of the E. T. semitone system of logarithms.

Let us now compute by this method the value of a perfect fifth, whose vibration ratio is $\frac{3}{2}$, expressed in E. T. semitones.

$$\frac{a-b}{a+b} = \frac{1}{5} = \cdot 2$$

$$\left(\frac{a-b}{a+b}\right)^3 = \frac{1}{5 \times 25} = \cdot 008$$

$$\left(\frac{a-b}{a+b}\right)^5 = \frac{1}{5 \times 25^2} = \cdot 000\ \ 32$$

$$\left(\frac{a-b}{a+b}\right)^7 = \frac{1}{5 \times 25^3} = \cdot 000\ \ 0128$$

Whence,

$$\left(\frac{a-b}{a+b}\right) = \cdot 2$$

$$\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3 = \cdot 002\ \ 666\dot{6}$$

$$\frac{1}{5}\left(\frac{a-b}{a+b}\right)^5 = \cdot 000\ \ 0640$$

$$\frac{1}{7}\left(\frac{a-b}{a+b}\right)^7 = \cdot 000\ \ 0018$$

$$\overline{\cdot 202\ \ 7325}$$

Multiplying this by the constant C,

$$\begin{array}{r} 34{\cdot}62469 \\ {\cdot}20\ 27325 \\ \hline 69\ 24938 \\ 692494 \\ 242373 \\ 10387 \\ 692 \\ 173 \\ \hline \end{array}$$

we have 7·0195499 as the value of the perfect fifth in E. T. semitones. It is within one unit in the seventh decimal place of the correct value,

7·019 5500

This is one of our fundamental data, procured by an independent process, in which we have not employed the labours of those who constructed the tables of common logarithms.

If however we consent to use these, we have a much more simple process available. It is easy to shew, by the theory of logarithms, that, in any two systems of logarithms, the logarithms of given numbers are proportional to one another; that is to say, any one system of logarithms can be transformed into any other by multiplication by some factor, which is called the modulus of transformation.

This relation is easily shewn to exist between E. T. semitones and any other system of logarithms, e.g. the common system.

For if we refer to the table at p. 83, and form a third column containing the common logarithms of $x, x^2, x^3, \ldots,$

then, if ξ be the logarithm of x

2ξ will be the logarithm of x^2

3ξ " " x^3

and so on;

and $\xi, 2\xi, 3\xi \ldots\ldots$

are obviously proportional to $1, 2, 3, \ldots\ldots\ldots\ldots$

which are the corresponding numbers of the E. T. system.

We have therefore only to find the factor by which common logarithms must be multiplied, to convert them into E. T. semitones.

This is easily done by the consideration that the common logarithm of 2, which is ·3010300, must become 12 when transformed.

The required factor is consequently,

$$\frac{12}{\cdot 3010300} = 39{\cdot}86314$$

It is only necessary to multiply the common logarithm of any ratio by this number, to get the equivalent in E. T. semitones.

We may, if we please, execute the multiplication directly; or we may divide by ·3010300, and multiply by 12, which is perhaps a little shorter. Or we may adopt the rules given in the note on p. 14, which perform the process more shortly, employing an arithmetical artifice.

When we have obtained, by any of these methods, the values of the fifth and third, all questions connected with the intervals of the diatonic scale can be solved by means of addition, subtraction, and multiplication and division by low numbers.

By means of the same values, the fifth and third, and intervals derived from their combination, can be compared with intervals formed by the division of the octave, with great facility.

Those who are acquainted with the use of common logarithms often employ them, instead of semitones, for these purposes; and it has been frequently proposed to use the division of the octave into 301 equal intervals, by means of which the common logarithmic tables read into the required division with considerable accuracy. It is necessary to remember however, in using such an approximation as this, that the solution of problems in beats (p. 17) generally requires five, and sometimes six, significant figures; any lower approximation, not having special properties with respect to the exact measuring of all intervals to be investigated, will be liable to error. The system of 301 in particular fails to represent the equal temperament altogether; its fifths are not particularly good; it does not admit of employment for demonstrating the difference between the different systems with approximately perfect fifths; nor can it be employed at all for the demonstration of the properties of systems of the mean-tone class. For these purposes at least five places of logarithms must be taken; and where the E. T. system has relations of any interest, the E. T. semitone is vastly superior as a unit to the logarithm.

There is an approximate method which has been occasionally employed, which it seems worth while to discuss with the view of obtaining some criterion of its accuracy.

The discussion will lead us, by analogy, to the deduction from equation (A) of a new approximate formula, of considerable accuracy for small intervals.

The old method consists in taking the difference between the vibration ratio and unity, and treating it as a measure of the interval.

Thus, in a major tone, whose ratio is

$$\frac{9}{8}, \quad \text{or} \quad 1\frac{1}{8},$$

$\frac{1}{8}$ would be taken as the measure of the interval.

We shall show that this is equivalent to taking the first term of a known logarithmic series, which is less convergent than the series in equation (A); i. e. would require the employment of more terms to get an accurate result.

The following series is proved in treatises on logarithms:—

$$(B) \ldots \log y = M\left\{(y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \ldots\right\}$$

The value of M is always half that of C in equation (A); so that for E. T. semitones,

$$M = 17{\cdot}31235$$

If we consider the major tone as $\frac{9}{8}$, then

$$y - 1 = \frac{1}{8}$$

$$(y-1)^2 = \frac{1}{8^2}$$

$$(y-1)^3 = \frac{1}{8^3}$$

and so on.

Again, if we take $\frac{8}{9}$, the descending ratio of the major tone, $1 - y = \frac{1}{9}$, and we can put the above series into the form,

$$(C) \ldots \log y = -M\left\{(1-y) + \frac{1}{2}(1-y)^2 + \frac{1}{3}(1-y)^3 + \ldots\right\}$$

The value is negative, indicating that the interval is taken downwards.

Here, in successive terms,

$$1 - y = \frac{1}{9}$$

$$(1-y)^2 = \frac{1}{9^2}$$

$$(1-y)^3 = \frac{1}{9^3}$$

and so on.

To compare these with equation (A).

If we consider the ratio $\frac{9}{8}$,

then $a = 9 \qquad b = 8$,

and
$$\frac{a-b}{a+b} = \frac{1}{17}$$
$$\left(\frac{a-b}{a+b}\right)^3 = \frac{1}{17^3}$$

and so on.

The terms with given index are less than those of (B) or (C); and there is only half the number of terms up to any given index, since the terms with even powers are not in series (A). Consequently by twice as many terms of either (B) or (C) we do not obtain the same accuracy as by any given number of terms of the series in (A), which we made use of above.

The second term of (B) is $\frac{1}{128}$

of (C) $\frac{1}{162}$;

these are of such magnitude that, in the present case (major tone), neither can be neglected in any computation which is intended to take count of such quantities as a comma.

The second term of (A), on the other hand,
$$= \frac{1}{3}\left(\frac{a-b}{a+b}\right)^3$$
$$= \frac{1}{3 \times 17^3}$$
$$= \frac{1}{14739}$$
a quantity which may be neglected for some purposes.

So long then as the approximation is restricted to small intervals, we may roughly compute E. T. equivalents by the approximate formula,
$$34{\cdot}6247 \times \frac{a-b}{a+b} = \text{number of E. T. semitones};$$
where $\frac{a}{b}$ is the vibration ratio, and 34·6247 the constant of equation (A).

For very small intervals the approximation will possess considerable accuracy.

Example. To compute approximately the value of a comma in E. T. semitones:

$$a = 81 \qquad\qquad b = 80$$

$$\frac{a-b}{a+b} = \frac{1}{161}$$

$$\frac{34{\cdot}6247}{161} = {\cdot}21506 \text{ E. T. semitones,}$$

which is correct to the fifth place of decimals.

On the other hand, if we compute a fifth, $\frac{3}{2}$, by this method, we get

$$\frac{34{\cdot}6247}{5} = 6{\cdot}9249 \text{ E. T. semitones.}$$

The correct value is 7·01955

error ·09465

and the error of the process amounts to about a tenth of a semitone, or nearly $\frac{1}{2}$ a comma.

Determining the octave, we get,

$$a = 2 \qquad\qquad b = 1$$

$$\frac{34{\cdot}6247}{3} = 11{\cdot}5412$$

with an error of about half a semitone.

To reverse the process.

Putting $\frac{a-b}{a+b} = z$, we find,

$$\frac{a}{b} = \frac{1+z}{1-z}.$$

If then we are given a small interval in semitones (x), we can find its ratio by this process.

For $\qquad x = 34{\cdot}6247 \left(\frac{a-b}{a+b}\right)$

(by approximate equation (A);)

whence $\qquad \frac{a-b}{a+b} = \frac{x}{34{\cdot}6247}$

$$= z$$

whence we can find the ratio, from

$$\frac{a}{b} = \frac{1+z}{1-z}.$$

Remembering that $z = \dfrac{x}{34 \cdot 6247}$,

we have,
$$\frac{a}{b} = \frac{34 \cdot 6247 + x}{34 \cdot 6247 - x}$$

This form may be sometimes useful for small intervals.

Example. To find the vibration ratio of one E. T. semitone. $(x=1)$

$$\frac{a}{b} = \frac{35 \cdot 6247}{33 \cdot 6247}$$

or, since the accuracy hardly reaches to the last figures,

$$= \frac{35 \cdot 62}{33 \cdot 62}$$

$$= \frac{17 \cdot 81}{16 \cdot 81} \text{ nearly,}$$

and the required ratio is a little greater than

$$\frac{18}{17}.$$