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# Well-formedness in two dimensions: a generalization of Carey and Clampitt's theorem†

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The famous results of Carey and Clampitt [N. Carey and D. Clampitt, *Aspects of well-formed scales*, Music Theory Spectr. 11 (1989), pp. 187–206] focus on scales generated by one interval and explain why some of these scales are preferable to others. Those preferable are called well-formed (WF). Their explanation is based on a theorem showing equivalence between 'symmetry' and 'closure'. In this paper, we propose and prove a generalization of this theorem. Instead of scales with a single generator, tone systems generated by two intervals are considered. In addition, various examples are given to illustrate the developed theoretical framework. Among them, the ancient Indian 22-*śruti* system is interpreted as a WF two-dimensional tone system generated by the fifth and the *śruti*. Finally, we draft open problems pertaining to the presented theory.

**Keywords:** generated scale; closure condition; symmetry condition; generic *Tonnetz*; generated tone system; tight; loose; well-formed



## 1. Introduction

Consider the following chain of tones generated by the perfect fifth:

$$\cdots \longrightarrow B_\flat \longrightarrow F \longrightarrow C \longrightarrow G \longrightarrow D \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow F_\sharp \longrightarrow \cdots$$

Five- or seven-note segments of this chain underlie ubiquitous tone systems: the anhemitonic pentatonic scale and the diatonic heptatonic scale. Why does six-note segment not form a 'good' scale?

There are several equivalent answers to this question. In their 1989 article [2], Carey and Clampitt presented two of them. They introduced two conditions – 'the closure condition' (CC) and 'the symmetry condition' (SC) – and showed that they are equivalent. They also coined the term 'well-formed' (WF) to name the scales that meet the two conditions.[1]

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The CC is more straightforward. It expresses the fact that the interval between the limiting points of the generation, i.e. between the starting tone and the tone following the last generated one (or, the last one and the one preceding the first one), does not contain any of the other tones. None of the tones of the pentatonic scale falls between $B_\flat$ and $A$, and similarly, none of the tones of the diatonic scale lies between $B_\flat$ and $B$. On the other hand, there are several tones of the 'hexatonic' fifth-generated scale that occupy a position within the interval $B_\flat$ and $E$ (or $E$ and $B_\flat$). We will say that the former scales are and the latter one is not *tight*.

The small[2] interval between the limiting points is called *comma*. Since there is no tone included within the comma, we may set the limiting points equivalent. This leads to an equivalence relation on the potentially infinite set of the generated tones (and their intervals). A symmetrical finite structure[3] arises from this equivalence relation. For example, in the case of the diatonic scale, the equivalence relation is given by $7\phi \equiv 0$, where $\phi$ denotes the perfect fifth. Both intervals $(C, G)$ and $(B, F)$, though generated differently as $\phi$ and $(-6\phi)$ and being very different in their pitch realizations, belong to the same equivalence class $[\phi]$, i.e. are considered structurally same.

The SC (or 'WF') reflects the relationship between the generation patterns of the intervals and their spans[4] in the pitch realization of the scale.[5] In a WF scale, any two intervals have equivalent generation patterns if and only if they have same spans. Consider the intervals $(C, E)$ and $(A, C)$ in the diatonic scale of $C$ major generated by the fifth. Their generation patterns $4\phi$ and $(-3\phi)$ are equivalent. At the same time, they both span two steps in the scale. If we, on the other hand, omitted the last generated tone and took the 'Pythagorean hexatonic' scale, the generation pattern $4\phi$ of the interval $(C, E)$ would be equivalent to the generation pattern $(-2\phi)$ of the interval $(D, C)$. However, their spans would be very different: 2 and 5, respectively.

Carey and Clampitt's theorem[6] states that the SC (i.e. well-formedness) and the CC (i.e. tightness) are equivalent.

THEOREM 1.1 (Carey and Clampitt) *Let S be a tone system generated by a single interval. Then S is tight if and only if it is WF.*

Our endeavour is to generalize this theorem for two dimensions. Let us illustrate our generalization on an example. Instead of a line of fifths, consider the *Tonnetz*[7] generated by both the fifth $\phi$ and the major third $\theta$ (Figure 1). In the diatonic system, two-thirds are equivalent to a fifth, i.e. $2\theta \equiv \phi$, which leads to the first comma $\kappa_1 = (-\phi + 2\theta)$. Similarly, adding a third to three-fifths results in a small interval. This gives the second comma $\kappa_2 = (-3\phi - \theta)$. The equivalence relation implied by the commas partitions the infinite set of tones of the *Tonnetz* into seven equivalence classes. One of them includes all the $D$'s positioned on the intersections of commas depicted as double dotted-lines in our diagrams. The seven classes are the nodes of an underlying symmetrical structure of the tone system. We will call this structure *generic Tonnetz*. Figure 2 shows the generic *Tonnetz* of this tone system.

Subsequently, we select a representative of each equivalence class. The selection of the representatives is depicted on our diagram by circles drawn around the tone letters. We always select representatives in accordance with the tiling implied by the commas. The tone $D$ plays a special role and we will call it *extremity*.[8] The selection of the representatives expresses how the tones are generated. For example, by selecting ${}_1B$, instead of $B$, we determined that the corresponding tone is generated as a fifth down and a third up, rather than three-fifths up from the extremity $D$. Finally, if we decide on the pitch of the generators (e.g. $\phi = \log_2(3/2)$ and $\theta = \log_2(5/4)$), we fully specify a *generated tone system* (GTS).[9]

A generalized version of the CC can be easily conceived and verified for this particular case: the pitches of all the inner tones lie outside the cluster of the pitches of the limits of the system. (We call *limits* the tones $D$, ${}^1D_\flat$, ${}_1D$, and ${}_2D_\sharp$.) The SC can also be introduced: the

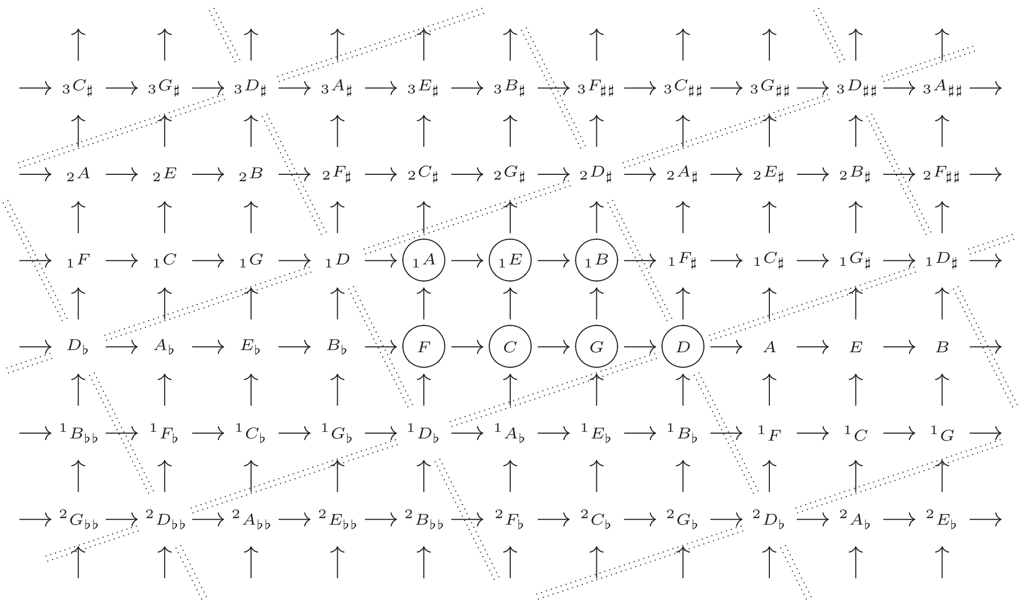



Figure 1. The diatonic system with two generators.

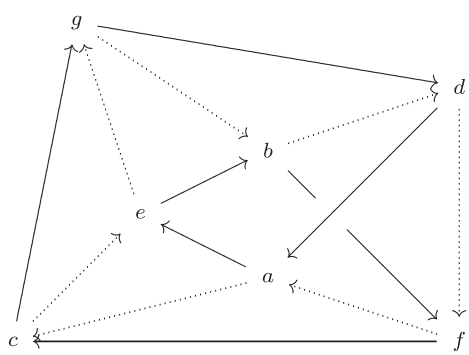



Figure 2. The diatonic system as a two-dimensional g-*Tonnetz*.

intervals given by the generic *Tonnetz* should be in a one-to-one relation to their spans. For example, consider the intervals $({}_1A, C)$ and $(C, {}_1E)$. They belong to the same equivalence class as $[\phi - \theta] = [\phi - \theta + \kappa_1] = [\theta]$. Similarly, the spans of the intervals are also the same; they are both equal to 2. It can be directly verified that this GTS is WF. It models the diatonic scale in just intonation.

## 2. Formal framework

In this section, we present technical details of the formal framework of the theory. The basic mathematical conventions assumed in the presentation are summarized in Appendix 1. In addition, Appendix 2 contains an extended example illustrating all main theoretical concepts introduced.

### 2.1. *Generic Tonnetz*

Definition 2.1 *Let $G = (T, I, \mathsf{int})$ be a commutative GIS and assume that group $I$ is generated by a finite subset $X$. A* generic Tonnetz (*g*-Tonnetz) *is the arrow-labelled directed graph $N = (T, A, X, \mathsf{int}^*)$, where $A$ denotes the pre-image of $X$ under* int, *i.e. $A = \mathsf{int}^{-1}[X]$, and* $\mathsf{int}^*$ *denotes the restriction of* int *to $A$, i.e. $\mathsf{int}^* = \mathsf{int}|A$. Further, we say that dimension of the g*-Tonnetz *$N$ is $n$ if $X$ has exactly $n$ distinct elements.*[10]

The asterisk will be omitted in the notation of $\mathsf{int}^*$ if there is no risk of confusion.

*Example 2.2* The model of the diatonic scale in just intonation described above can be expressed within our formal framework in the following way. The seven tones of the GIS are $T = \{c, d, e, f, g, a, b\}$ and the group of intervals is isomorphic to seven-element cyclic group $I \cong \mathbf{Z}_7$. The int function assigns the usual diatonic interval to any pair of tone letters, e.g. $\mathsf{int}(c, e) = 2$. The set of generators comprises the fifth and the third: $X = \{2, 4\}$. Then, the set of arrows of the g-*Tonnetz* comprises all pairs of tones corresponding to the intervals 2 and 4: $A = \{(c, e), (d, f), \ldots, (c, g), (d, a), \ldots\}$. The resulting g-*Tonnetz* is shown in Figure 2. Mazzola [10] describes a structure which is somewhat dual to this g-*Tonnetz* and calls it a 'harmonic strip'.[11]

The arrow labelling $\mathsf{int}^*$ distinguishes the third-arrows from the fifth-arrows. In Figure 2, the dotted lines denote third arrows and the regular ones the fifth arrows. Later, we will usually omit this explicit distinction if the labelling is clear from the context.

Lemma 2.3 *Consider two commutative GISs $G_1 = (T_1, I_1, \mathsf{int}_1)$ and $G_2 = (T_2, I_2, \mathsf{int}_2)$. Further assume that $I_i$ is generated by its finite subset $X_i$ and $N_i = (T_i, \mathsf{int}_i^{-1}[X_i], X_i, \mathsf{int}_i)$ are the related g*-Tonnetze, *for $i = 1, 2$. Then the following conditions are equivalent.*

(1) *There is a group isomorphism between $I_1$ and $I_2$ mapping $X_1$ onto $X_2$.*
(2) *There is a GIS-isomorphism between $G_1$ and $G_2$ mapping $X_1$ onto $X_2$.*
(3) *$N_1$ and $N_2$ are isomorphic.*

Lemma 2.4 *Consider an abelian group $I$ generated by its subset $X$. Then there exist a set $T$ and a mapping $\mathsf{int} : T \times T \to I$ such that $G = (T, I, \mathsf{int})$ is a commutative GIS. This GIS is unique, up to isomorphism. As a consequence, the g*-Tonnetz *$N(I; X) = (T, \mathsf{int}^{-1}[X], X, \mathsf{int})$ also exists and it is, up to isomorphism, unique.*

The two previous lemmas[12] imply that, up to isomorphism, a GIS is determined by the underlying group of intervals and a g-*Tonnetz* by the underlying group of intervals *and* the selected set of generators of the group. Therefore, a complete study of the abelian groups and their generating subsets is sufficient for the complete understanding of the g-*Tonnetze*.

For our study of the g-*Tonnetze*, we will rely on presentations of abelian groups as quotient groups of free abelian groups.

Lemma 2.5 *Let an abelian group $I$ be generated by the finite set $X$ of $n$ elements. Then, there exists a subset $K$ of $\mathbf{Z}[X]$ with $m$ elements where $m \leq n$ and $\mathbf{Z}[X]/\langle K\rangle$ is a presentation of $I$. If $I$ is finite, then $m = n$.*

Definition 2.6 *Let $X = \{\xi_1, \ldots, \xi_n\}$ be a set, $K = \{\kappa_1, \ldots, \kappa_m\}$ be a subset of $\mathbf{Z}[X]$ and $n \leq m$. Assume the g-Tonnetz $N = N(\mathbf{Z}[X]/\langle K\rangle; X)$ from Lemma 2.4. We say that the elements $\kappa_i$, $1 \leq i \leq m$ are commas and $K$ is a set of commas of $N$. The g-Tonnetz $N$ is denoted as $\mathsf{gT}(X|K)$.*

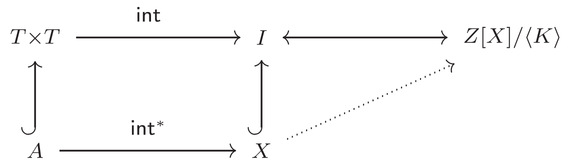



Figure 3. Diagram of mappings related to a g-*Tonnetz*.

According to Lemma 2.5, the assumptions of the previous definition cover, up to isomorphism, any g-*Tonnetz* and we may limit our investigation of the g-*Tonnetze* to those of this type. In other words, any g-*Tonnetz* is fully determined, up to isomorphism, by the set of generators and a set of commas. Figure 3 shows the mappings related to a g-*Tonnetz*. The arrows with curved tails denote the natural injections of the subsets $A$ and $X$ in their supersets $T \times T$ and $I$, respectively.

### 2.2. *Definition of GTS*

DEFINITION 2.7 *Let $T$ and $X$ be finite non-empty sets, $K$ be a subset of the free abelian group $\mathbf{Z}[X]$ freely generated by $X$, and let both $K$ and $X$ have exactly $n$ elements. An $n$-dimensional GTS is a sextuple $(T, X, K, \mathsf{int}, \mathsf{spec}, \mathsf{pitch})$ where the following conditions are satisfied.*

(1) $\mathsf{int} : T \times T \to \mathbf{Z}[X]/\langle K\rangle$ *is a mapping with the following properties*:
  (i) $\mathsf{int}(t, u)\mathsf{int}(u, v) = \mathsf{int}(t, v)$ *for all* $t, u, v \in T$.
  (ii) *For any* $t \in T$ *and* $\alpha \in \mathbf{Z}[X]/\langle K\rangle$, *there is a unique* $u \in T$ *such that* $\mathsf{int}(t, u) = \alpha$.
(2) $\mathsf{spec} : T \to \mathbf{Z}[X]$ *is a mapping with the following property*:
  (i) $[\mathsf{spec}(t)] + \mathsf{int}(t, u) = [\mathsf{spec}(u)]$, *for any* $t, u \in T$.
(3) $\mathsf{pitch} : X \to \mathbf{R}/\mathbf{Z}$ *is a mapping*.

*The elements of* $T, X, K$, *and* $I = \mathbf{Z}[X]/\langle K\rangle$ *are called* tones, generators, commas, *and* intervals, *respectively. The mappings* int, spec, *and* pitch *are called interval function*, *specifying function*, *and pitch function*, *respectively.*

The first condition assures that $(T, I, \mathsf{int})$ is a commutative GIS. In fact, this definition extends the concept of g-*Tonnetz*. Let $e$ denote the canonic projection from $\mathbf{Z}[X]$ to $\mathbf{Z}[X]/\langle K\rangle$, i.e. $e$: $\mathbf{Z}[X] \to \mathbf{Z}[X]/\langle K\rangle$, $\xi \mapsto [\xi]$. Then $(T, \mathsf{int}^{-1}[e[X]], e[X], \mathsf{int})$ is a g-*Tonnetz*. A GTS differs from a g-*Tonnetz* in that it specifies how the tones are generated from the generators and how the generators are tuned. While the g-*Tonnetz* is a perfectly symmetrical structure, the specifying function and the pitch function bring the 'imperfectness' into the picture. This is illustrated in the following example.

*Example 2.8* Consider the Pythagorean heptatonic (e.g. $C$ major scale). The underlying g-*Tonnetz* is a graph consisting of seven nodes connected by arrows on a circle of fifths. In this graph, there is no difference between the perfect fifth $(C, G)$ and the diminished fifth $(B, F)$, both are depicted as fifth arrows. If the information about the specifying function and the pitch function is added, the asymmetry between the two kinds of fifth becomes explicit.

### 2.3. *Comma-demarcated GTS*

For a given g-*Tonnetz*, the tones may be generated (i.e. the specifying function can be defined) in many ways. However, only some of them are of real interest. In the present paper, a basic requirement is that the values of the specifying function are 'demarcated' within $\mathbf{Z}[X]$ by

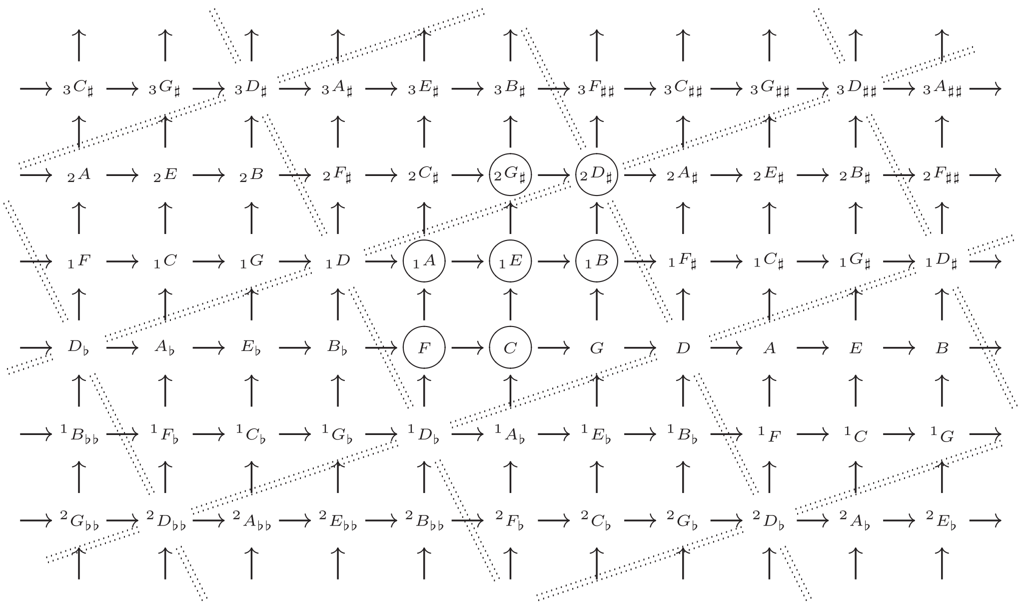


Figure 4. A non-demarcated GTS.

the commas. The demarcatedness, introduced in the next definition, means that we take a block $P(K)$ of elements of $\mathbf{Z}[X]$ within a parallelogram[13] delimited by the selected set of commas.[14]

Definition 2.9 *Assume an n-dimensional GTS S with a set of commas* $K = \{\kappa_1, \ldots, \kappa_n\}$. *Denote*:

$$P(K) = \left\{ \zeta \in \mathbf{Z}[X] | \zeta = \sum_{\kappa \in K} r(\kappa)\kappa,\ r(\kappa) \in [0, 1) \right\}$$

*We say that S is comma-demarcated* (*or*, *simply*, *demarcated*) *if there is a tone* $e \in T$ *such that* $\mathsf{spec}[T] = \mathsf{spec}(e) + P(K)$. *Further*, *we say that*:

(1) *e is the* extremity *of S*;
(2) $t \in T$ *is an* edge tone (*or*, *more precisely*, *a* $\kappa$-edge tone) *if* $\mathsf{spec}(t) = \mathsf{spec}(e) + r\kappa$, *for some* $\kappa \in K$ *and* $r \in [0, 1)$;
(3) $t \in T$ *is an inner tone if* $\mathsf{spec}(t) = \mathsf{spec}(e) + \sum_{\kappa \in K} r(\kappa)\kappa$ *and* $r(\kappa) \neq 0$ *for all* $\kappa \in K$;
(4) *the elements* $\mathsf{spec}(e) + \sum_{\kappa \in K} r(\kappa)\kappa \in \mathbf{Z}[X]$ *for* $r(\kappa) \in \{0, 1\}$ *are called limits of S*.[15]

*Example 2.10* The GTS shown in Figure 1 is demarcated as are most of the other GTSs discussed later. Figure 4 illustrates a GTS which is non-demarcated, although musically very interesting. It models the Hungarian scale ($A$ minor or $E$ major). Moreover, not only this selection of commas fails to provide a demarcated GTS. In general, it is not possible to select the commas in such a way that the resulting demarcated GTS would model the Hungarian scale.[16]

*Example 2.11* Figure 5 shows two of the possible interpretations of the 12-tone chromatic scale as a demarcated GTS. In both cases, there are some edge tones besides the extremity: $D$ in the first and $C$ and ${}_1E$ in the second approach. Note also that the underlying g-*Tonnetz* of these GTSs are same as the subgroups generated by the sets of commas are equal.

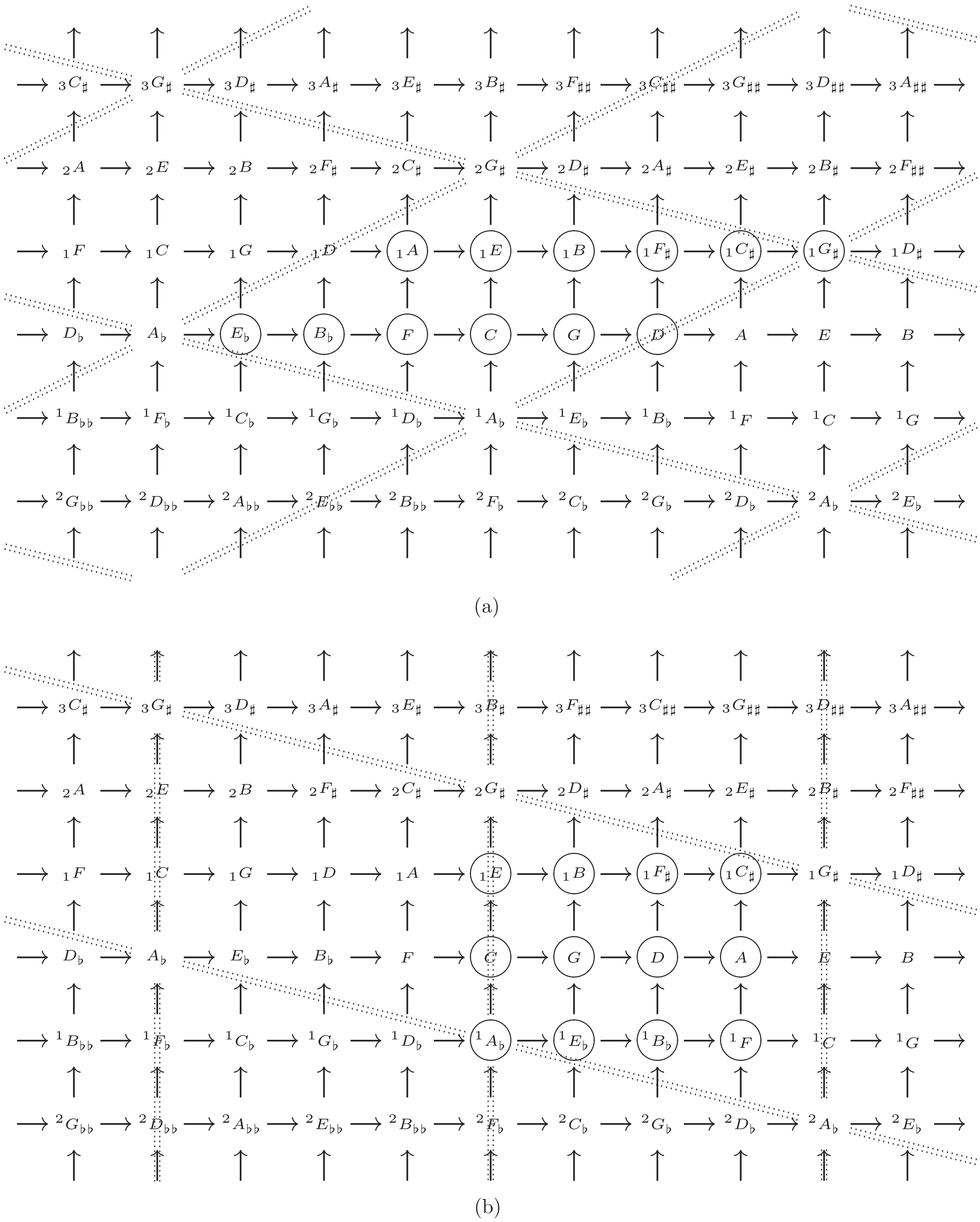



Figure 5. Chromatic scale as a two-dimensional GTS – two approaches.

*Example 2.12* The extremity plays a special role in the GTS; it participates the most on the asymmetries of the pitch realization of the underlying g-*Tonnetz*. Consider the one-dimensional fifth-generated Pythagorean diatonic scale with the limits $B_\flat$ and $B$ and with the extremity at $B$. The same system of tones can be interpreted as a GTS having the limits $F$ and $F_\sharp$ and the extremity at $F$. The tones $F$ and $B$, the extremities of these two GTSs, are in a sense the most problematic ones because their pitch realizations make up the dissonant tritone. On the other hand, the tone $D$

is placed in the centre of the system, positioned furthest from the dissonant pair of tones $(F, B)$.[17] This reflects the common understanding of the tone $D$ as the central tone.

*Example 2.13* However, the position of the tone $D$ is very different in the diatonic scale in just intonation shown in Figure 1. There it takes the role of the extremity. This is in line with the well-known problem of the just intonation: $D$ participates on the impure fifth $(D, {}_1A)$ problematizing the $D$ minor triad. Sometimes, the problem is mitigated by replacing $D$ with ${}_1D$ (which improves $D$ minor but worsens $G$ major). On the other hand, the central positions are occupied by the tones $C$ and $E$. This corresponds with the central role of the $C$ major triad and its dual $A$ minor triad.[18]

I dare to hypothesize that the considerations drafted in the two previous examples may explain why the Dorian was the most preferred and most frequent modus in the modal repertoire during the times ruled by the Pythagorean tuning and why the emphasis moved towards the major–minor system of the later period when the just intonation and its approximations emerged.

## 2.4. *Transposed GTSs and neighbouring GTSs*

Consider the $C$ major and $F$ major scales in the Pythagorean tuning. There are two ways of modelling their relation within our theoretical framework. First, we may say that $F$ major is the GTS with limits $B_\flat$ and $B$ and the extremity at $B_\flat$, and similarly, $C$ major is the GTS with limits $F$ and $F_\sharp$ with the extremity at $F$ (Figure 6(a) and (b)). This way, $C$ major is a *transposition* of $F$ major. However, there is another way of looking at the relation between the two tone systems: $C$ major can also be modelled as a GTS with the same limits as $F$ major, i.e. $B_\flat$ and $B$, but having the extremity positioned at $B$ (Figure 6(c)). In that case, we say that the GTSs are *neighbouring*.

As another example of neighbouring GTSs (in a two-dimensional case) consider the GTSs of the diatonic scale with extremities at $D$ or at ${}_1D$. (We mentioned the diatonic system with extremity ${}_1D$ in Example 2.13 as a possible approach to mitigate the issue with the impure fifth $(D, {}_1A)$.) These GTSs are neighbouring, as well.[19] In a neighbouring GTS, some of the commas are replaced by their opposites and the specifying function is changed correspondingly. The following lemma

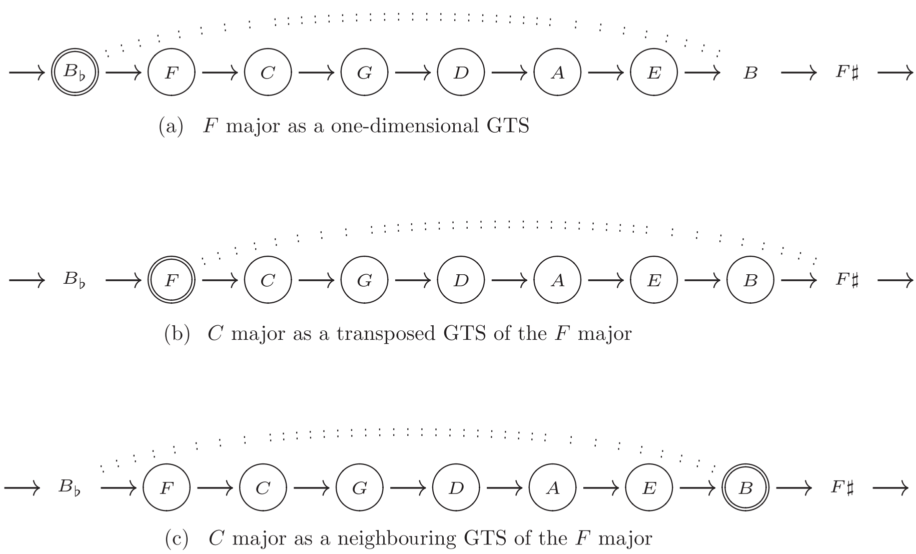



Figure 6. Transposed and neighbouring GTSs.

deals with the technical details of the construction of neighbouring GTSs. The formal definition follows.

LEMMA 2.14 *Let* $S = (T, X, K, \mathsf{int}, \mathsf{spec}, \mathsf{pitch})$ *be a demarcated GTS with the extremity* $e \in T$ *and let* $N : K \to \{-1, 1\}$ *be a mapping. Assume the following conditions.*

(1) $K_N = \{N(\kappa)\kappa | \kappa \in K\}$.
(2) *A mapping* $\mathsf{spec}_N : T \to \mathbf{Z}[X]$ *is defined in the following way. For any* $t \in T$, $\mathsf{spec}(t) - \mathsf{spec}(e) = \sum_{\kappa \in K} r_t(\kappa)\kappa$, *consider the set* $K(t) = \{\kappa \in K | r_t(\kappa) = 0, N(\kappa) = -1\}$. *The mapping* $\mathsf{spec}_N$ *assigns the value* $\mathsf{spec}_N(t) = \mathsf{spec}(t) + \sum_{\kappa \in K(t)} \kappa$.

*Then* $S_N = (T, X, K_N, \mathsf{int}, \mathsf{spec}_N, \mathsf{pitch})$ *is a demarcated GTS.*

*Proof* The first condition of Definition 2.7 follows from the fact that $\mathbf{Z}[X]/\langle K \rangle = \mathbf{Z}[X]/\langle K_N \rangle$. As any $\kappa \in K$ is in the same class as 0, we have that $[\mathsf{spec}_N(t)] = [\mathsf{spec}(t)]$. This implies the second condition of the definition. Therefore, $S_N$ is a GTS. The demarcatedness results from the following. For any tone $t$, $\mathsf{spec}(t)$ is not in $\mathsf{spec}_N(e) + P(K_N)$ if and only if there are some $\kappa_i$'s for which $r_t(\kappa_i) = 0$ and $N(\kappa_i) = -1$. The mapping $\mathsf{spec}_N$ makes the required corrections by the $\kappa_i$'s so that $\mathsf{spec}_N(t)$ belongs to $\mathsf{spec}_N(e) + P(K_N)$. ■

DEFINITION 2.15 *Assume the notation from Lemma 2.14. The GTSs* $S$ *and* $S_N$ *are called* neighbouring.

It is a little more straightforward to define the transposition formally. The next definition gives the details. Note that it does not require the GTS to be demarcated to define its transposition. However, it is easy to see that if a GTS is demarcated, then all its transpositions are demarcated.

DEFINITION 2.16 *Let* $S = (T, X, K, \mathsf{int}, \mathsf{spec}, \mathsf{pitch})$ *be a GTS and* $\zeta$ *be any element of the group* $\mathbf{Z}[X]$. *Consider the mapping defined as* $\mathsf{spec}_\zeta(t) = \mathsf{spec}(t) + \zeta$ *for all* $t \in T$. *Then we say that the GTS* $S_\zeta = (T, X, K, \mathsf{int}, \mathsf{spec}_\zeta, \mathsf{pitch})$ *is a* transposition (*or, more specifically, the* $\zeta$-transposition) *of* $S$.

### 2.5. *Pitch-related properties*: size *and* span

In Definition 2.7, the pitch function was required to be defined for the generators only. However, it follows from the basic properties of free groups that there is a unique group homomorphism $\mathsf{pitch}^*: \mathbf{Z}[X] \to \mathbf{R}/\mathbf{Z}$ such that $\mathsf{pitch}(\xi) = \mathsf{pitch}^*(\xi)$ for all $\xi \in X$. In notating this homomorphism, we omit the asterisk and also call it a pitch function if there is no risk of confusion.

DEFINITION 2.17 *Consider two elements* $\alpha, \beta \in \mathbf{Z}[X]$. *The size of the ordered pair* $(\alpha, \beta)$ *is the number* $r$, $0 \le r < 1$, *for which* $\mathsf{pitch}(\alpha) \oplus r = \mathsf{pitch}(\beta)$, *i.e.* $r = \mathsf{pitch}(\beta - \alpha)$. *We denote the size of* $(\alpha, \beta)$ *by* $\mathsf{size}(\alpha, \beta)$. *Further, if* $\alpha = \mathsf{spec}(a)$ *and* $\beta = \mathsf{spec}(b)$ *for tones* $a, b \in T$, *we also say that the size of the pair of tones* $(a, b)$ *is* $r$ *and write* $\mathsf{size}(a, b) = r$.

DEFINITION 2.18 *Assume a GTS* $S$. *We define a ternary relation 'between' on* $\mathbf{Z}[X]$. *Let* $\alpha_1, \alpha_2, \alpha_3$ *be elements of* $\mathbf{Z}[X]$. *We say that* $\alpha_2$ *is between* $\alpha_1$ *and* $\alpha_3$ *and write* $\lhd(\alpha_1, \alpha_2, \alpha_3)$ *if the following condition holds in* $\mathbf{R}$.

$$\mathsf{size}(\alpha_1, \alpha_2) + \mathsf{size}(\alpha_2, \alpha_3) = \mathsf{size}(\alpha_1, \alpha_3)$$

*Further, if* $\alpha_i = \mathsf{spec}(t_i)$ *for* $t_i \in T$ *and* $i = 1, 2, 3$, *then we also say that the tone* $t_2$ *is between the tones* $t_1$ *and* $t_3$ *and write* $\lhd(t_1, t_2, t_3)$.

The regular addition of real numbers in the condition from the previous definition cannot be replaced by the addition modulo 1. Informally, tone $u$ is between the tones $t$ and $v$ if it follows $t$ and is followed by $v$ (clockwise) on the standard clock-face representation of the pitch domain $\mathbf{R}/\mathbf{Z}$. Notice that, for three distinct tones, $u$ is between $t$ and $v$ if and only if it is not between $v$ and $t$. The relation $\lhd$ enables us to define the span of a pair of tones.

DEFINITION 2.19 *Assume a GTS $S$ and consider two tones $t, u \in T$. We say that the* span *of the ordered pair $(t, u)$ is $(k-1)$ if there are exactly $k$ distinct tones between $t$ and $u$. We denote the span of $(t, u)$ by* $\mathsf{span}(t, u)$. *Further, the ordered pair $(t, u)$ is called a* step *if* $\mathsf{span}(t, u) = 1$.

## 2.6. *SC and CC*

DEFINITION 2.20 (CC) *Consider a demarcated GTS $S$ with the set of commas $K$ and the extremity $e$. We say that*:

(1) *$S$ is* loose *if for some $\kappa \in K$ there are two tones $m, n \in T$ such that the following conditions hold simultaneously*:

$$\lhd\, (\mathsf{spec}(e), \mathsf{spec}(m), \mathsf{spec}(e) + \kappa)$$

$$\lhd\, (\mathsf{spec}(e) + \kappa, \mathsf{spec}(n), \mathsf{spec}(e))$$

(2) *$S$ is* tight *if it is not loose*.

In a loose GTS, there is a tone between two limits whose distance is a comma, from both sides of the pitch circle. This is illustrated in Figure 7. Therefore, the comma is not sufficiently small. The Pythagorean system is loose, for instance, for six tones and it is tight for five or seven tones. In the case of two-dimensional GTSs, the diatonic system from Figure 1 and both chromatic ones from Figure 5 are tight. An example of a loose two-dimensional demarcated GTS is easy to make up; one such is shown in Figure 8: $G$ is between $F$ and ${}^1B_\flat$ and $D$ is between ${}^1B_\flat$ and $F$.

The definition of the SC involves one complication; all the neighbouring GTSs must be considered. The reason is that, in general, the span is not invariant for neighbouring GTSs. Therefore, a GTS is *semi-WF* if its span function is in a one-to-one correspondence with the interval function and it is *WF* if, in addition, the span function is invariant for the neighbouring GTSs.

DEFINITION 2.21 (SC) *Consider a demarcated GTS $S$. We say that*:

(1) *$S$ is semi-WF if for all tones $t_1, u_1, t_2, u_2 \in T$*:

$$\mathsf{int}(t_1, u_1) = \mathsf{int}(t_2, u_2) \implies \mathsf{span}(t_1, u_1) = \mathsf{span}(t_2, u_2)$$

(2) *$S$ is WF if it is semi-WF and for any mapping $N\colon K \to \{-1, 1\}$ and all tones $t, u \in T$ it holds that* $\mathsf{span}(t, u) = \mathsf{span}_N(t, u)$, *where* $\mathsf{span}_N$ *denotes the span function in the neighbouring GTS $S_N$*.

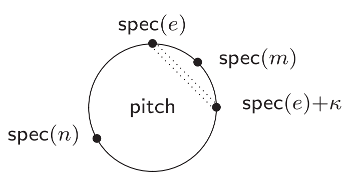



Figure 7. Pitch representation of loose GTS.

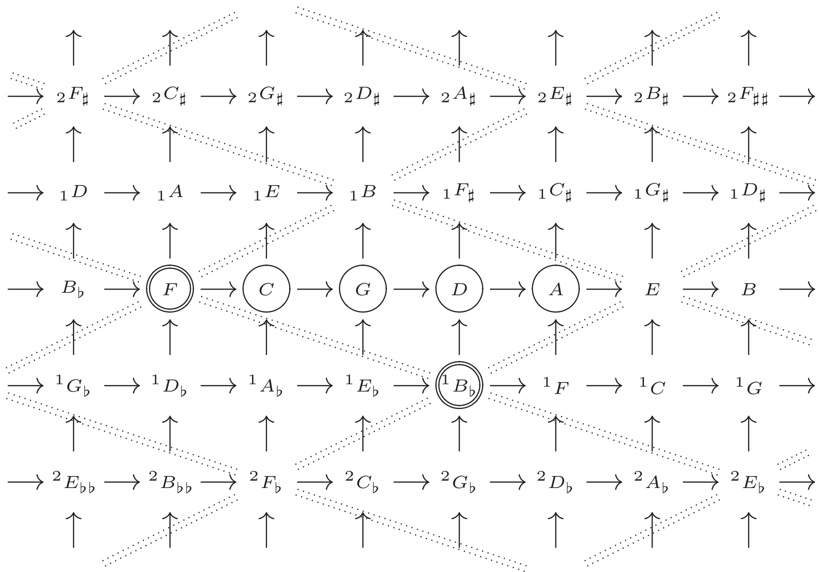


Figure 8. A semi-WF and non-WF GTS.

*Example 2.22* We give an example of a GTS which is semi-WF and non-WF. Figure 8 visualizes two neighbouring GTSs. In the first one, the extremity is at $F$. This GTS is semi-WF. The span of any fifth is 3 and the span of any third is 4. On the other hand, the neighbouring GTS where the extremity is at ${}^1B_\flat$ is not semi-WF. The span of the fifth $(C, G)$ is 2 and the span of the fifth $(G, D)$ is 4. Therefore, the span function is not invariant for neighbouring GTSs and the GTSs are not WF.

## 3. The main theorem

We are ready to state the main result of the paper. It asserts that a two-dimensional demarcated GTS with inner tone(s) is WF if and only if it is tight.

THEOREM 3.1 (Generalized Carey–Clampitt's Theorem) *Let S be a demarcated two-dimensional GTS with at least one inner tone. Then S is tight if and only if it is well-formed.*

*Proof* See Appendix 3. ■

As mentioned before, this theorem can be considered a generalization of the famous results of Carey and Clampitt [4]. They formulated a 'closure condition' (CC) and a 'symmetry condition' (SC) for the category of one-dimensional generated scales and concluded that these two conditions are equivalent. They also introduced the term 'WF scales' referring to the scales meeting the two equivalent conditions.

In the present approach,[20] the category of two-dimensional GTS is considered. The CC is generalized through the property of being 'tight' as defined in Definition 2.20. The situation with the SC is more complex. Carey and Clampitt's SC can be expressed in several different versions which are equivalent for the one-dimensional case. In our generalization, we consider the following version. The intervals of same generation orders have same scale step orders. (In particular, every generating interval is of the same span.) The generation order of intervals is

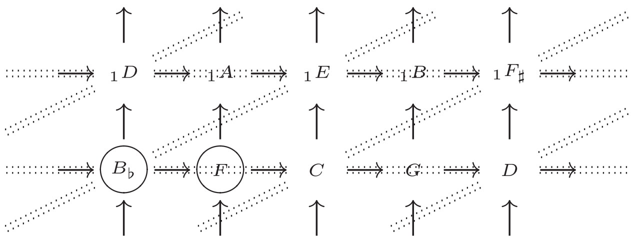



Figure 9. A demarcated GTS without inner nodes contradicting the equivalence of the symmetry and CC.

generalized in the multi-dimensional case through the int function (Section 2.1 and Definition 2.7). The scale step order is generalized through the span function, which depends on the pitch function (Definitions 2.19 and 2.7). This way Carey and Clampitt's SC is generalized here by the property of being 'WF' as defined in Definition 2.21.

There is another important point in which the theorem generalizes Carey and Clampitt's work. In the one-dimensional case, the property of well-formedness is a criterion for identifying structurally integral (or 'symmetric') scales, which – if symmetry and structural integrity are preferred – may be considered a criterion for discriminating preferable scales. The scales generated by the perfect fifth are WF if they comprise certain number of tones: 2, 3, 5, 7, 12, 17, 29, 41, 53, etc. Carey and Clampitt used the theory of continued fractions, which enabled them to give exact formulas for these numbers.[21] Although the present theory does not generalize the concept of continued fractions to two-dimensional systems,[22] it still gives a valid criterion for selecting structurally integral GTSs. It can be easily checked whether a pair of commas determines a tight GTS. If it does the theorem asserts that the resulting GTS is WF. From this point of view, it is not any more true that it is up to 'the theorist to determine *a priori* which commas are theoretically useful' [20, Note 6, p. 63]. The theorem provides an objective way for selecting sets of commas determining theoretically useful two-dimensional GTSs with symmetrical structure.

*Example 3.2* Theorem 3.1 assumes existence of at least one inner tone.[23] In this example, we show that this assumption cannot be omitted. Consider the demarcated GTS shown in Figure 9 with limits $B\flat, C, {}_1F\sharp$, and ${}_1E$. It can be directly verified that this GTS is not tight. On the other hand, a generating fifth in any neighbouring GTS has always span 1 and so does any generating third, as well. Therefore, this GTS is WF and contradicts the implication $WF \rightarrow Closure$.[24]

## 4. Examples

The concepts of g-*Tonnetz* and GTS apply to surprisingly many phenomena encountered in various musical contexts. They are suitable to model situations where two (or more) basic elements are freely combined to built complex, symmetrical structures. As a basic example, the diatonic and chromatic scales in just intonation are WF GTSs with two generators: the perfect fifth and the pure major third (Figures 1 and 5). In a similar way, the anhemitonic pentatonic scale can be modelled as a WF two-dimensional GTS, as well.

However, the generating intervals need not necessarily be the fifth and the third. For example, to model the pitch helix, a model of tonal space known from the psychology of hearing, we need to consider for the generators the octave $o$ and the semitone $\sigma$, and one comma $12\sigma - o$. The infinite g-*Tonnetz* $\mathsf{gT}(o, \sigma | 12\sigma - o)$ models the pitch helix. (The pitch function in a corresponding GTS would assign values from $\mathbf{R}$ rather then $\mathbf{R}/\mathbf{Z}$.)

Other WF GTSs of musical interest can be conceived within the usual chromatic universe of 12 tones. For instance, the octatonic and the hexatonic scales can be interpreted as GTSs with two generators: a step $\sigma$ and a third $\theta$. In both systems, we have the comma $\kappa = 2\sigma - \theta$ reflecting the idea that two steps give a third. The other commas are $\lambda_1 = 3\theta$ and $\lambda_2 = 4\theta$ in the hexatonic and the octatonic scale, respectively. This is in accordance with the periodicity of the third in either system. If $\mathsf{pitch}(\sigma) = 1/12$ and $\mathsf{pitch}(\theta) = 4/12$, we obtain a GTS with the g-*Tonnetz* $\mathsf{gT}(\sigma, \theta|\kappa, \lambda_1)$, which is WF and models the hexatonic system. On the other hand, $\mathsf{pitch}(\sigma) = 1/12$ and $\mathsf{pitch}(\theta) = 3/12$ leads to another GTS with the g-*Tonnetz* $\mathsf{gT}(\sigma, \theta|\kappa, \lambda_2)$, which is also WF and models the octatonic system (see the first two figures in Appendix 4).

As still another GTS within the chromatic space, consider the g-*Tonnetz* generated by semitone $\sigma$ and tritone $\tau$ with commas $2\tau$ and $6\sigma + \tau$. The resulting GTS contains all 12 tones and is WF. The underlying g-*Tonnetz* can be drawn on a Möbius strip (see the last figure in Appendix 4). It may have analytical application for those pieces of the twentieth-century repertoire which make use of the 'most-dissonant' intervals instead of the most consonant ones. In terms of the dissonance, this g-*Tonnetz* is a kind of conceptual opposite of the standard *Tonnetz*. At the same time, it is an interesting coincidence that it shares certain structural singularity with the diatonic system (Figure 2), which also can be drawn on a Möbius strip.

Finally, we want to focus on the systems where the generating elements are the perfect fifth and a small interval of the size approximately a half of semitone. These generators are important for various music cultures, notably for Arabic and Indian music. In the Indian music theory, the small interval is usually called *śruti* and we will use this name. So we consider a g-*Tonnetz* with two generators $\phi$ (the fifth) and $\sigma$ (the *śruti*). The basic problem is to specify the commas.

One comma is easy to think of. When we move from a given point by $\phi$ in opposite directions, we arrive to points a whole tone apart (considering the octave equivalence, of course). Now, if we bend the lower tone upwards by two *śrutis* and the upper one by the same amount downwards, we obtain almost the same tone.[25] This is the basis of the first comma: $-\phi + 2\sigma \equiv \phi - 2\sigma$, which gives the comma $\kappa = 4\sigma - 2\phi$.

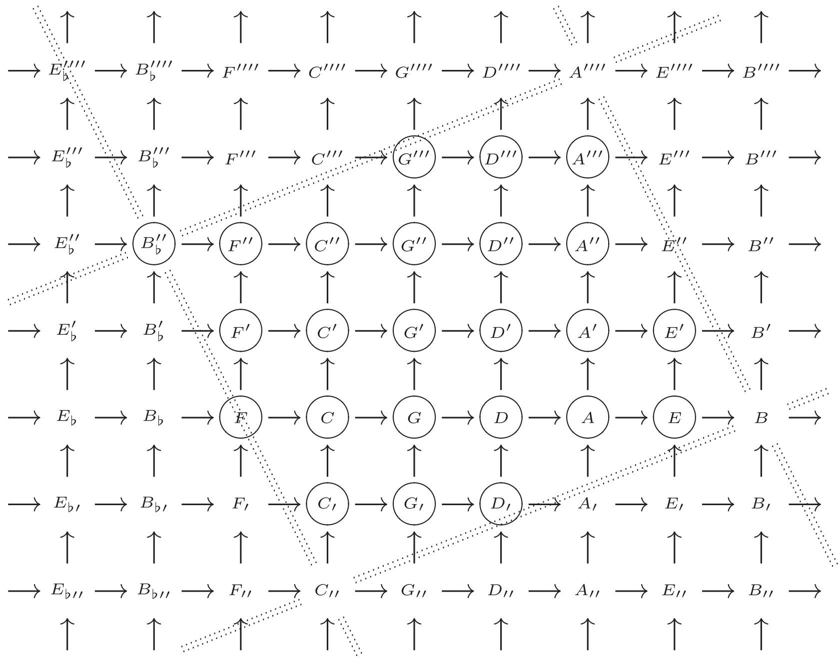
The other comma is related to the one underlying the Pythagorean pentatonic. A tone tuned as the fifth perfect fifth is lower than the starting tone just by a small interval. By bending the fifth fifth upwards results in a comma. However, there is an issue: should it be bent by two or by one *śruti*? In the first case, the other comma is $\lambda_1 = 5\phi + 2\sigma$. In the second case, it is $\lambda_2 = 5\phi + 1\sigma$. It is fascinating that both options seem to have been (unconsciously) applied by major music cultures – the Arabic and the Indian. Figure 10 shows the GTSs given by the g-*Tonnetze* $\mathsf{gT}(\phi, \sigma|\kappa, \lambda_1)$ and $\mathsf{gT}(\phi, \sigma|\kappa, \lambda_2)$. Both GTSs are WF.

The first solution leads to a 24-tone WF GTS. Arabic music theory knows a system of 24 small intervals called *nīms*. It is usually explained as a result of splitting each tone of the 12-tone chromatic system into two quarter tones. Our approach provides an alternative explanation for the structure of the system. In this explanation, the *nīms* do not have to be (acoustically) uniform.[26]

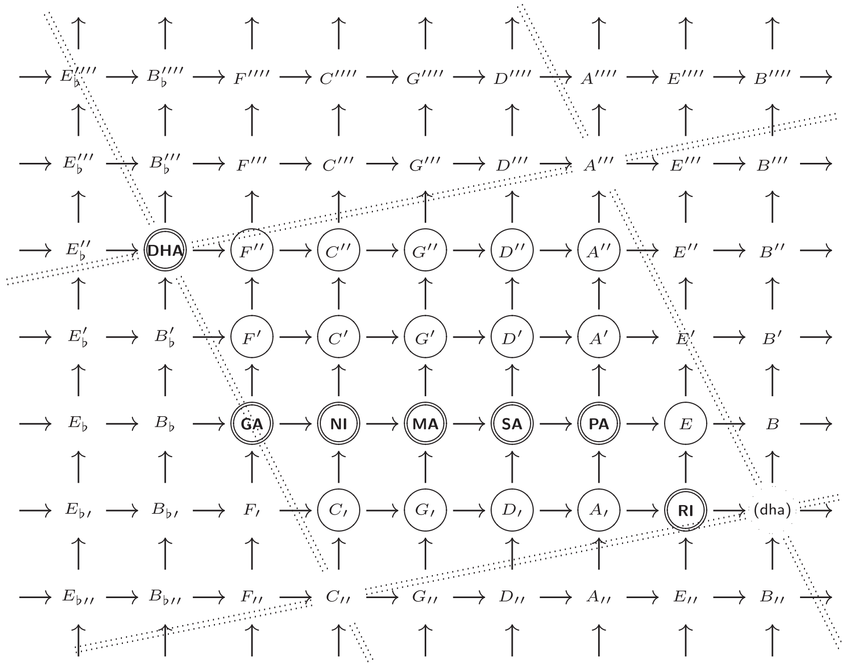
More striking is the fact that the WF GTS implied by the second set of commas comprises 22 elements. It seems to model suitably the Indian system of 22 *śrutis*. There is no generally accepted explanation for the number of 22 in this system.[27] Our explanation of this number is very simple and surprisingly accurate. It only relies on four basic assumptions:

(1) the perfect fifth and the *śruti* are basal;
(2) a fifth down and two *śrutis* up equals approximately a fifth up and two *śrutis* down;
(3) five-fifths up equals approximately one *śruti* below;
(4) the resulting system is symmetrical (WF).

Notice that we did not have to specify the exact value of a *śruti*. It is sufficient that it is *approximately* a half of semitone. Then both resulting GTSs are tight and WF. The present theory aims at investigating structural properties of tone systems rather than addressing tuning issues and details of acoustical realizations of them.




(a) The GTS of 24 *nīms*.




(b) The GTS of 22 *śrutis* with highlighted *sa-grāma*.

Figure 10. The GTSs generated by the perfect fifth and the *śruti*.

## 5. Open problems

PROBLEM 5.1 (Carey and Clampitt's theorem for higher dimensions) *The formal framework developed in Section* 2 *is not confined to two dimensions. It is therefore natural to ask whether a generalization of Carey and Clampitt's theorem for higher dimensions holds or not. I conjecture that it does.*

PROBLEM 5.2 (Generalization of Myhill's property) *In their theory of diatonicism* [22]*, Clough and Myerson introduced the concept of 'generic' and 'specific' intervals and showed that Myhill's property* (*MP*)*, among other equivalent conditions, is a key structural feature of the diatonic scales. MP states that every generic interval appears in exactly two sizes. Carey and Clampitt* [23] *proved that, in non-degenerate one-dimensional generated scales, MP is equivalent to WF. It means that any one-dimensional generated scale is WF if and only if it has up to two different sizes for any (generic) interval.*

*In our approach, 'generic intervals' correspond to 'intervals'* $\mathsf{int}(a, b)$ (*i.e. all elements of the group* $\mathbf{Z}[X]/\langle K\rangle$)*, 'specific intervals' correspond to values* $\mathsf{spec}(b) - \mathsf{spec}(a)$ (*i.e. some elements of* $\mathbf{Z}[X]$)*, and 'sizes' of intervals are given by values* $\mathsf{size}(a, b)$ (*i.e. elements of* $\mathbf{R}/\mathbf{Z}$)*. Therefore, if pitch values of generators are not immensurable, different specific intervals may have same sizes. However, this does not happen for tight demarcated GTSs. But still, there does not seem to be a straightforward generalization of MP for the two-dimensional case. As can be easily shown a necessary condition similar to MP exists: in a demarcated WF two-dimensional GTS any generic interval appears in up to four different sizes. This condition is not sufficient, though. The problem is to find a necessary and sufficient condition to WF, which could be considered a generalization of MP for two-dimensional GTSs.*

PROBLEM 5.3 (Hellegouarch's condition) *Hellegouarch* [14,15] *proposed a theory of 'natural scales' which is in certain aspects closely related to the present theory. Among other matters, he investigated the free abelian group of rank* 3 (*i.e. generated by three generators*: *the octave, the perfect fifth and the major third*) *and its quotient groups modulo its subgroups generated by two commas. He found a sufficient condition for such quotient groups to be cyclic. This result became a basis for his exploration of 'improvements' of the Pythagorean scales.*

*Let us briefly rephrase Hellegouarch's approach within our framework. As he does not consider the octave equivalence and the octave is one of the generators, his quotient groups of rank* 3 *correspond to the two-dimensional GTSs. Consider a GTS with a set of generators X consisting of the perfect fifth* $\phi$ *and the major third* $\theta$*, i.e.* $X = \{\phi, \theta\}$*, and a set of commas* $K = \{\kappa_1, \kappa_2\}$*. Take* $b_i, c_i \in \mathbf{Z}$*,* $i = 1, 2$ *such that* $\kappa_i = b_i\phi + c_i\theta_i$ *and* $a_i \in \mathbf{Z}$ *as the appropriate coefficients for octaves, i.e.* $a_i$ *minimizes the number* $|a_i + b_i\mathsf{pitch}(\phi_i) + c_i\mathsf{pitch}(\theta_i)|$*. Hellegouarch's condition* (*HC*) *can be put in the following way.*

$$\exists\, x, y, z \in \mathbf{Z} : \quad \left\|\begin{matrix} x & a_1 & a_2 \\ y & b_1 & b_2 \\ z & c_1 & c_2 \end{matrix}\right| = 1. \qquad \text{(HC)}$$

*As per Hellegouarch's theorem, HC implies that* $\mathbf{Z}[X]/\langle K\rangle$ *is cyclic.*

*It can be easily verified that HC* $\not\Rightarrow$ *WF. It suffices to consider the first Hellegouarch's example in the third table on page 16 of* [15]*:* $\kappa_1 = -3\phi, \kappa_2 = \theta$*. The corresponding GTS fulfils HC but is loose* (*i.e. not WF*)*.*

*On the other hand, I hypothesize*[28] *that WF* $\Rightarrow$ *HC for two-dimensional*[29] *GTSs. If this hypothesis is valid, then it has two interesting implications. First, all WF two-dimensional GTSs have a cyclic group of intervals* $\mathbf{Z}[X]/\langle K\rangle$*. Second, well-formedness is a more restrictive condition for selecting* *theoretically interesting generated scales than HC. At the same time, it can be argued that the CC (equivalent to WF) is a basic cognitive requirement. Therefore, WF would seem to be a more accurate criterion for exploring two-dimensional generated scales.*

PROBLEM 5.4 (Fokker's periodicity blocks) *Fokker's 'periodicity blocks'* [12,13] *are selections from the free abelian group (harmonic lattice) related to what is called here specifying function. Fokker's selections are also given by a parallelogram (or a parallelepiped) delimited by a set of commas. In this sense, periodicity blocks correspond to GTSs. However, not all periodicity blocks are 'demarcated' as defined in the present paper. Loosely speaking, Fokker does not require the vertices of the delimiting parallelogram to belong to the free abelian group. Therefore, the concept of periodicity block is more general than the concept of comma-demarcated GTS. The Hungarian scale given above as an example of non-demarcated GTS (Example* 2.10) *can be modelled as a periodicity block.*

*These considerations lead to a basic question: does the equivalence of the SC and the CC hold also in the more general case of Fokker's periodicity blocks? To be correct, before putting this question, one should generalize the definitions of SC and CC, which required the GTS to be demarcated. So, the problem is to generalize SC and CC for periodicity blocks and to investigate for which periodicity blocks these conditions are equivalent.*


## Acknowledgements

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## Notes

1. For a related, independently formulated concept of 'the moments of symmetry,' see also [3].
2. 'Small' does not necessarily mean the smallest in the system. For example, the size of the comma $(B\flat, B)$ is larger than the size of the inner interval $(E, F)$ in the Pythagorean diatonic system.
3. It may be formalized as a finite Lewinian GIS. For a definition of GIS, see [4] or Appendix 1 of the present paper.
4. Informally, the term 'span' can be defined as the number of steps between two notes in a scale ordered by pitch. For example, the span of $(C, E)$ is 2 in the $C$ major scale. A formal definition will follow.
5. In the one-dimensional case, there are several equivalent ways to put the SC. Here, we select one which is easy to generalize for the multidimensional context.
6. It was first published without proof in [2]. A proof can be found in Carey's dissertation [5].
7. The term 'Tonnetz' is commonly used today. For an overview of the historical development of this concept, refer to [6]. The work of Richard Cohn [7, 8] played a key role in its revival in the modern music theory. In the present paper, we propose one possible mathematical formalization of the concept.
8. The next section contains the formal definition.
9. Figure 1 and the similar ones visualize two-dimensional GTS's (refer also to Definition 2.7) on a usual *Tonnetz*, which corresponds to the free abelian group $\mathbf{Z}[\phi, \theta]$ generated by the perfect fifth $\phi$ and the pure major third $\theta$. The double-dotted lines represent its subgroup generated by the set of commas $K$. The circled positions visualize the selection of representatives of the quotient group $\mathbf{Z}[\phi, \theta]/\langle K \rangle$, i.e. the specifying function. The tone letters express the values of the elements of $\mathbf{Z}[X]$ under the pitch function. (The number-prefixes track the syntonic commas: e.g. ${}_1E$ means that the tone mapped to this position is tuned by one syntonic comma lower than the tone mapped to the position denoted by $E$. The syntonic comma can be expressed as $4\phi - \theta$ and it belongs to the subgroup $\langle K \rangle$ generated by the set of commas in this example.) The corresponding generic *Tonnetz* can be drawn on a torus: take the parallelogram delimited by the commas (the double-dotted lines) containing the selected positions and glue its opposite edges together and the circles and nodes make up the generic *Tonnetz*. Its nodes are the tones of the GTS and the values of the interval function is implied by the arrows. In certain singular cases, the generic *Tonnetz* may be drawn on simpler 'canvas' such as Möbius strip or circle. The one corresponding to the GTS visualized by this

figure may be drawn on the Möbius strip. See also Appendix 2, which elaborates the mathematical details of this example.

10. Although g-Tonnetz is a graph, the term 'dimension of g-Tonnetz' as defined here differs from the concept of 'dimension of a graph' as used elsewhere [9].
11. Mazzola's 'harmonic strip' is the geometric nerve of a set of seven three-element sets corresponding to the triads of a diatonic scale. The 0-simplices (nodes) represent the triads, the 1-simplices (edges) represent the pairs of triads with common tone(s), and the 2-simplices (triangles) represent the triples of triads with a common tone. The harmonic strip can be visualized on a Möbius strip.
12. We give the lemmas in this section without proofs. The reason is that they state known facts and/or their proofs are straightforward. In particular, the first two lemmas imply that the categories of groups and GISs are equivalent in the commutative case. Kolman [11] gave a detailed account showing that this equivalence holds in general (i.e. even without the assumption of commutativity).
13. The formal framework presented in this section is not confined to two dimensions. Strictly speaking, $P(K)$ is contained in a parallelogram only in the two-dimensional case. In the three-dimensional case, it is a parallelepiped, etc.
14. The concept of the demarcated GTS closely corresponds with that of 'periodicity block' of Fokker [12,13]. Fokker introduced this concept and heuristically explored certain particular periodicity blocks, which, in our terms, are GTSs generated by pure intervals. He did not state an objective criterion explaining his selection of the examined periodicity blocks. In the present paper, the main point is to show that the SC and the CC are equivalent for demarcated GTSs with inner tone(s). This may become a basis for defining such an objective criterion. In this sense, the present theory extends Fokker's work on periodicity blocks from the 1960's. See also the last problem discussed in Section 5.
15. The tone names in the subsequent figures stand for elements of the free group $\mathbf{Z}[X]$ (and their values under the pitch function). They should not be confused with the elements of $T$. See also Note 4 for more details about the graphing technique and Appendix 2 for a detailed explanation of an example.
16. However, the Hungarian scale is a 'periodicity block' as introduced by Fokker [12,13]. See also the open problem appertaining to the relation between GTSs and periodicity blocks drafted in Section 5.
17. In fact, the rules of strict counterpoint allowed to introduce the tritone $(B, F)$ if the tone $D$ was in bass. That way, $(D, B)$ and $(D, F)$ were considered acceptable intervals. In other words, the 'central' tone $D$ could mitigate the extremities $F$ and $B$.
18. Riemann's notation of minor triads would perfectly fit into this context. In his theories, the $A$ minor triad is considered the minor triad of the tone $E$. So, the 'central' tones $C$ and $E$ correspond with the Riemannian $C$ major and '$E$ minor'.
19. Taking the other two limits as the extremities leads to uncommon tone systems. These systems are structurally more complex in that they comprise four different step intervals (for a definition of a step interval, see below) while the common systems of just intonation mentioned in the text have step intervals only of three different kinds. However, a formal treatment of these considerations is out of scope of this paper.
20. A comparable approach to generalizing Carey and Clampitt's results can be considered the work of Hellegouarch [14,15].
21. For a recent work discussing the applications of continued fractions, see [16]. The apparatus of the continued fractions has been used in the theory of tuning for centuries. Although Douthett and Krantz assert that Drobisch [17] 'was the first to approximate musical intervals using continued fractions', (p. 48) it was probably Euler who first applied this approach to the study of tone systems in his famous *Tentamen novae theoriae musicae* [18, p. 260], published in 1739.
22. There have been various attempts to generalize the continued fractions to higher dimensions. One of the first ones was Barbour's [19] work.
23. In the previous version of this paper [1], an even stricter assumption was considered: the 'normalness' of the GTS. It is easy to see that a normal GTS, as defined there, always contains at least one inner tone. Therefore, the version of the theorem given in the present paper is more general.
24. On the other hand, the implication Closure → WF holds for demarcated two-dimensional GTSs in general, without the assumption of existence of an inner node. This follows directly from the proof of the theorem presented in Appendix 3.
25. This condition fits with the condition that a *śruti* is approximately a half of semitone in the following way. Assume that two *śrutis* are equal to a (chromatic) semitone. Then if we start from $C$, a fifth down and two *śrutis* up is $F\sharp$ and a fifth up and two *śrutis* down is $G\flat$. So the condition corresponds with the statement that a *śruti* is approximately a half of semitone as far as $F\sharp$ approximately equals $G\flat$.
26. The uniformity of *nīms* was the core of the strong arguments at the famous Congress of Cairo in 1932 about the acceptability of 24-tone equal temperament.
27. As an example from the recent mathematical music theory, Clough *et al.* [21] investigated this system. However, they did not address the question of the total number of *śrutis*. Possible explanations of the *śruti*-system as a two-dimensional GTS generated by perfect fifth and major third or as a three-dimensional GTS generated by perfect fifth, major third and natural seventh can be found in [12,13].
28. The idea of a proof (to be verified). Consider the free abelian group $\mathbf{Z}[o, \phi, \theta]$ and its elements $\kappa_1' = a_1 o + b_1 \phi + c_1 \theta$, $\kappa_2' = a_2 o + b_2 \phi + c_2 \theta$, and $\sigma = xo + y\phi + z\theta$. The equation with the determinant can be geometrically interpreted that there is no element of $\mathbf{Z}[o, \phi, \theta]$ contained in the parallelepiped delimited by $\kappa_1'$, $\kappa_2'$, and $\sigma$, except

for the vertices. Assume that HC does not hold and take for $\sigma$ any of the steps. Then we will obtain an element in the parallelepiped. This element will counterdict the CC.

29. This may even be true for one-dimensional GTSs. Hellegouarch restricts his investigation of one-dimensional GTSs to those which correspond to the convergents of $\log_2(3/2)$. Therefore, the GTSs corresponding to the semi-convergents (which according to the results of Carey and Clampitt are also WF) do not appear on Hellegouarch's list. However, it does not mean that they do not fulfil a one-dimensional version of HC. As an example, consider the Pythagorean diatonic scale. It is WF and is not listed by Hellegouarch. And it also satisfies HC because for $(x, y) = (-1, 2)$ we have:
$$\left\| \begin{matrix} x & -4 \\ y & 7 \end{matrix} \right\| = 1.$$



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## Appendix 1. Basic mathematical conventions

The letters **R** and **Z** denote the set of real numbers and the set of integers, respectively. $\mathbf{Z}_n$ denotes the additive cyclic group of $n$ elements $\{0, 1, \ldots, n-1\}$. We assume octave equivalence and we use the arithmetics modulo 1 for the domain of pitch. Addition modulo 1 is distinguished from regular addition by using the symbol '$\oplus$'.

A Cartesian product of sets $S_1, \ldots, S_n$ is a set $P = S_1 \times \cdots \times S_n$ of ordered $n$-tuples $(s_1, \ldots, s_n)$ such that $s_i \in S_i$ for $i = 1, \ldots, n$. If all the sets $S_i$ are same, i.e. $S_i = S$ for $i = 1, \ldots, n$, we write $P = S^n$. An $n$-ary *relation* on a set $S$ is a subset of the Cartesian product $S^n$, i.e. a set of ordered $n$-tuples of elements of $S$. Therefore, a ternary relation on

$S$ is a set of ordered triples of elements of $S$. An equivalence relation on a set $S$ is a binary relation which is reflexive, symmetric, and transitive. An equivalence relation implies a partitioning of the set $S$ into equivalence classes.

The *mappings*, which are binary relations, are notated in a usual way: $m\colon A \to B$ denotes a mapping $m$ with domain $A$ and codomain $B$, $m\colon a \mapsto b$ or $m(a) = b$ means that $m$ maps the element $a \in A$ to its image $b \in B$, and $m[A]$ denotes the image of the domain as a subset of the codomain. The pre-image of an element $b_0 \in m[A]$ is the element $a_0$ such that $m(a_0) = b_0$. In this case, we write $m^{-1}(b_0) = a_0$. Analogically, for any subset $X \subset m[A]$, we denote $m^{-1}[X]$ the pre-image of $X$ under $m$, i.e. $m^{-1}[X]$ is the set of all elements $a \in A$ such that their image $m(a)$ is in $X$. A restriction of the mapping $m$ to a subset $A' \subset A$ of the domain is the mapping $m'\colon A' \to B$ defined by $m'(a') = m(a')$ for all $a' \in A'$. This restriction is notated as $m' = m|A$. The mapping $m$ is in this case also called an extension of $m'$.

The concepts of *group*, group homomorphism, group isomorphism, normal group, and group equivalence relation are used in a usual way. We also make an extensive use of the concept of free abelian groups. A free (abelian) group $F(X)$ freely generated by the set of generators $X$ is a (abelian) group such that for any (abelian) group $G$ and any mapping $m\colon X \to G$, there exists a group homorphism $m^*\colon F(X) \to G$ which is an extension of $m$. It can be shown that for any non-empty set $X$, there exists a free (abelian) group $F(X)$ and it is unique, up to isomorphism. It is a well-known fact that any (abelian) group $I$ generated by a subset $X$ can be presented as a quotient group of the free (abelian) group $F(X)$. Assume a subset $K \subseteq F(X)$ of the free (abelian) group $F(X)$ and let $\langle K\rangle$ be the smallest normal subgroup of $F(X)$ containing all elements of $K$. We denote $F(X)/\langle K\rangle$ the quotient group of $F(X)$ modulo $\langle K\rangle$. If the group $I$ is isomorphic to $F(X)/\langle K\rangle$ we say that $F(X)/\langle K\rangle$ is a *presentation* of the group $I$. We denote $[\alpha]_K$ the equivalence class of an element $\alpha \in F(X)$ in $F(X)/\langle K\rangle$. If there is no risk of confusion, we omit the subscript '$K$'. All these ideas also apply in the commutative case as shown by the parenthetical adjective 'abelian' used throughout this paragraph. Any abelian group $I$ may be uniquely represented as a quotient group of the free abelian group $\mathbf{Z}[X]$ where $\mathbf{Z}[X] = \{\sum_{i=1}^{n} k_i\xi_i | k_i \in \mathbf{Z}, \xi_i \in X\}$. We follow a convention that the elements of $\mathbf{Z}[X]$ are notated by greek alphabet through the entire article.

The proposed theory relies on David Lewin's concept of the generalized interval systems (GIS) [4]. A GIS is a triple $(S, I, \mathsf{int})$, where $S$ is a set, $I$ is a group of intervals and $\mathsf{int}$ is a mapping assigning an interval to every ordered pair of elements of $S$ with the following properties:

(1) $\mathsf{int}(t, u)\,\mathsf{int}(u, v) = \mathsf{int}(t, v)$ for all $t, u, v \in S$.
(2) For any $t \in S$ and $\alpha \in I$ there is a unique $u \in S$ such that $\mathsf{int}(t, u) = \alpha$.

We will limit our investigation to commutative GISs, i.e. those for which the group of intervals is abelian. The notion of isomorphism can be defined for the category of GISs in a common way.

A basic concept of this paper – the generic *Tonnetz* – is defined as a directed graph. (In fact, the generic *Tonnetz* is a concept analogous to the concept of the 'Cayley graph' known from the geometrical theory of groups. From this point of view, the theory of the generic *Tonnetze* belongs to that branch of group theory.) We say that an ordered quadruple $(N, A, L_A, l_A)$ is an *arrow-labelled directed graph* if the following conditions hold.

(1) $N$ is a non-empty set of *nodes*;
(2) $A$ is a set of ordered pairs of nodes (i.e. a subset of the direct product $N \times N$) and its elements are called *arrows*;
(3) $L_A$ is a non-empty set of *arrow labels*;
(4) $l_A\colon A \to L_A$ is a mapping assigning arrow labels to arrows and is called an *arrow labelling*. If $l(a) = \alpha$ we say that $a$ is an $\alpha$*-arrow*.

The concept of graph isomorphism (isography) can be defined in a usual way.

## Appendix 2. A detailed example

In this appendix, we elaborate mathematical details for the first example of a two-dimensional GTS from the main text. It models the diatonic scale in just intonation and is visualized on Figure 1. Figure A1 reproduces this visualization.

### A.1. *GTS: basic notions*

The GTS comprises seven tones and is generated by the perfect fifth $\phi$ and the major third $\theta$. The following list specifies the concepts from Definition 2.7.

*Tones*: $T = \{c, d, e, f, g, a, b\}$.
*Generators*: $X = \{\phi, \theta\}$.
*Commas*: $K = \{\kappa_1 = -\phi + 2\theta, \kappa_2 = -3\phi - \theta\}$.
*Group of intervals*: $I = \mathbf{Z}[\phi, \theta]/\langle\kappa_1, \kappa_2\rangle$
This group is isomorphic to the cyclic group $\mathbf{Z}_7$.
*Specifying function*: We arbitrarily select the value of $d$ as 0. The value of all the other tones is implied by this selection.

$$\begin{aligned} \mathsf{spec}\colon c &\longmapsto -2\phi \\ d &\longmapsto 0 \\ e &\longmapsto -2\phi + \theta \end{aligned}$$

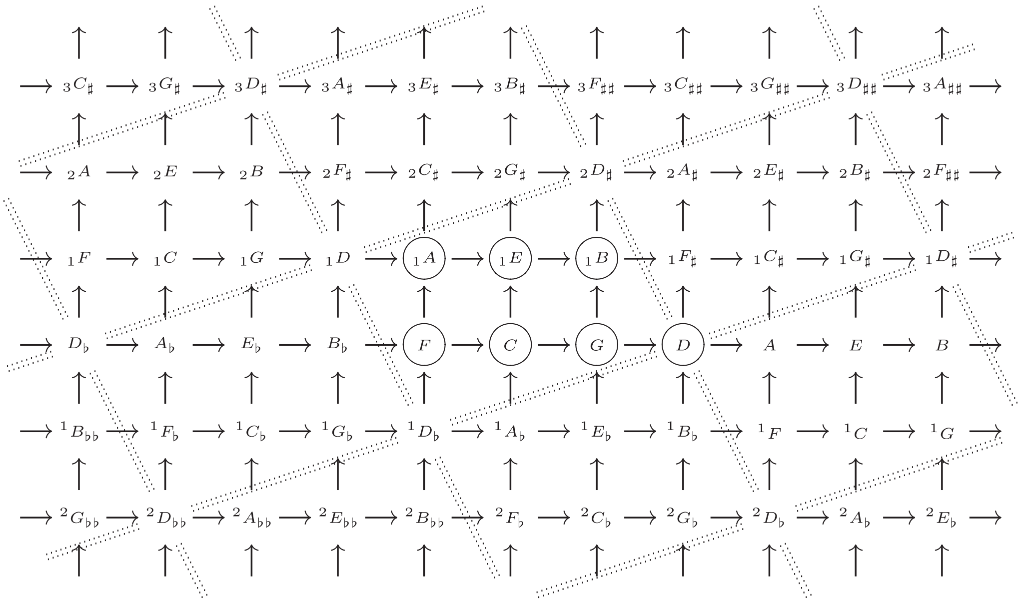



Figure A1. GTS $S$ modelling the diatonic scale in just intonation.

$$f \longmapsto -3\phi$$
$$g \longmapsto -\phi$$
$$a \longmapsto -3\phi + \theta$$
$$b \longmapsto -\phi + \theta.$$

*Interval function*: int: $(t_1, t_2) \mapsto [\mathsf{spec}(t_2)] - [\mathsf{spec}(t_1)]$.
*Pitch function*: pitch: $\phi \mapsto \log_2(\frac{3}{2}) = 0.585\ldots, \quad \theta \mapsto \log_2(\frac{5}{4}) = 0.322\ldots$
*GTS*: $S = (T, X, K, \mathsf{int}, \mathsf{spec}, \mathsf{pitch})$ is a GTS

## A.2. *Generic* Tonnetz

Every GTS determines a generic *Tonnetz*. Example 2.2 describes a g-*Tonnetz* related to our GTS $S$ and Figure 2 shows the graph. The group of intervals can be specified more precisely as $\mathbf{Z}[X]/\langle K\rangle$. The following list refers to Definition 2.1 and gives the details of the concepts defined there. (We consider the canonic projection $e\colon \mathbf{Z}[X] \to \mathbf{Z}[X]/\langle K\rangle, \xi \mapsto [\xi]$.)

*GIS*: $(T, \mathbf{Z}[X]/\langle K\rangle, \mathsf{int})$.
*Set of generators*: $e[X] = \{[\phi], [\theta]\}$.
*Nodes*: $T = \{c, d, e, f, g, a, b\}$.
*Arrows*: $A = \mathsf{int}^{-1}[e[X]] = \mathsf{int}^{-1}([\phi]) \cup \mathsf{int}^{-1}([\theta])$.
*Arrow labels*: $[\phi]$ and $[\theta]$.
$[\phi]$-*arrows*: $\mathsf{int}^{-1}([\phi]) = \{(c, g), (d, a), (e, b), (f, c), (g, d), (a, e), (b, f)\}$.
$[\theta]$-*arrows*: $\mathsf{int}^{-1}([\theta]) = \{(c, e), (d, f), (e, g), (f, a), (g, b), (a, c), (b, d)\}$.

## A.3. *Demarcatedness*

The set $P(K)$ from the definition of demarcatedness contains seven elements. It can be directly verified that the following equations hold:

$$\mathsf{spec}(c) = \mathsf{spec}(d) + \frac{2}{7}\kappa_1 + \frac{4}{7}\kappa_2,$$
$$\mathsf{spec}(d) = \mathsf{spec}(d) + \frac{0}{7}\kappa_1 + \frac{0}{7}\kappa_2,$$

$$\mathsf{spec}(e) = \mathsf{spec}(d) + \frac{5}{7}\kappa_1 + \frac{3}{7}\kappa_2,$$
$$\mathsf{spec}(f) = \mathsf{spec}(d) + \frac{3}{7}\kappa_1 + \frac{6}{7}\kappa_2,$$
$$\mathsf{spec}(g) = \mathsf{spec}(d) + \frac{1}{7}\kappa_1 + \frac{2}{7}\kappa_2,$$
$$\mathsf{spec}(a) = \mathsf{spec}(d) + \frac{6}{7}\kappa_1 + \frac{5}{7}\kappa_2,$$
$$\mathsf{spec}(b) = \mathsf{spec}(d) + \frac{4}{7}\kappa_1 + \frac{1}{7}\kappa_2.$$



Therefore, we have that $\mathsf{spec}[T] = \mathsf{spec}(d) + P(K)$ and the investigated GTS is demarcated. The following list contains details of the concepts from Definition 2.9.

*Extremity*: Tone $d$.
*Edge tones*: The only edge tone is the extremity $d$, which is both a $\kappa_1$-edge tone and a $\kappa_2$-edge tone.
*Inner tones*: All the tones except for the extremity are inner.
*Limits*: The following four elements of $\mathbf{Z}[\phi, \theta]$ are the limits:

$$\lambda_1 = \mathsf{spec}(d) = 0$$
$$\lambda_2 = \mathsf{spec}(d) + \kappa_1 = -\phi + 2\theta$$
$$\lambda_3 = \mathsf{spec}(d) + \kappa_2 = -3\phi - \theta$$
$$\lambda_4 = \mathsf{spec}(d) + \kappa_1 + \kappa_2 = -4\phi + \theta.$$

## A.4. *Neighbouring GTS*

Consider the following four mappings:

$$N^{++}: K \longrightarrow \{-1, 1\},\ \kappa_1 \longmapsto 1,\ \kappa_2 \longmapsto 1,$$
$$N^{-+}: K \longrightarrow \{-1, 1\},\ \kappa_1 \longmapsto -1,\ \kappa_2 \longmapsto 1,$$
$$N^{+-}: K \longrightarrow \{-1, 1\},\ \kappa_1 \longmapsto 1,\ \kappa_2 \longmapsto -1,$$
$$N^{--}: K \longrightarrow \{-1, 1\},\ \kappa_1 \longmapsto -1,\ \kappa_2 \longmapsto -1.$$

They determine four neighbouring GTSs, the first one being the GTS $S$. Figure A2 visualizes the neighbouring GTS $S^{--}$ determined by $N^{--}$. The following list gives the details of the GTS.

*Set of commas*: In this neighbouring GTS, both commas are reversed.

$$K^{--} = \{-\kappa_1 = \phi - 2\theta, -\kappa_2 = 3\phi + \theta\}$$

*Specifying function*: $K(t)$ is empty for every tone $t \neq d$ and $K(d) = K$. Therefore, the specifying function $\mathsf{spec}^{--}$ takes a different value only for $d$.

$$\begin{aligned} \mathsf{spec}^{--}\colon c &\longmapsto -2\phi \\ d &\longmapsto \kappa_1 + \kappa_2 = -4\phi + \theta \\ e &\longmapsto -2\phi + \theta \\ f &\longmapsto -3\phi \\ g &\longmapsto -\phi \\ a &\longmapsto -3\phi + \theta \\ b &\longmapsto -\phi + \theta \end{aligned}$$

*Neighbouring GTS*: $S^{--} = (T, X, K^{--}, \mathsf{int}, \mathsf{spec}^{--}, \mathsf{pitch})$.

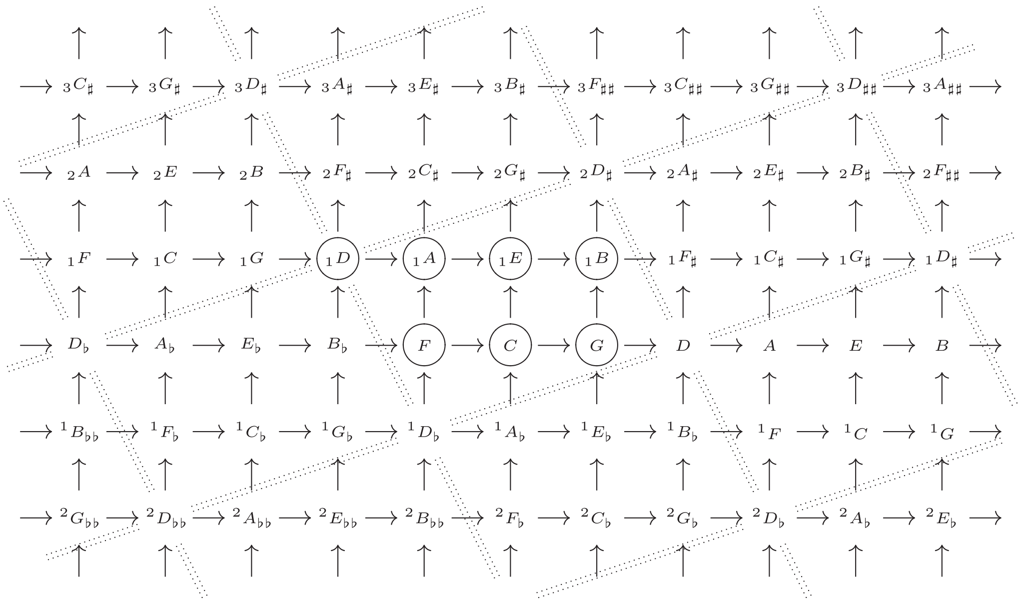


Figure A2. The neighbouring GTS $S^{--}$.

*Demarcatedness*: The demarcatedness of $S$ implies the demarcatedness of $S^{--}$. The following equations show the calculations:

$$\mathsf{spec}^{--}(c) = \mathsf{spec}^{--}(d) + \frac{5}{7}(-\kappa_1) + \frac{3}{7}(-\kappa_2)$$
$$\mathsf{spec}^{--}(d) = \mathsf{spec}^{--}(d) + \frac{0}{7}(-\kappa_1) + \frac{0}{7}(-\kappa_2)$$
$$\mathsf{spec}^{--}(e) = \mathsf{spec}^{--}(d) + \frac{2}{7}(-\kappa_1) + \frac{4}{7}(-\kappa_2)$$
$$\mathsf{spec}^{--}(f) = \mathsf{spec}^{--}(d) + \frac{4}{7}(-\kappa_1) + \frac{1}{7}(-\kappa_2)$$
$$\mathsf{spec}^{--}(g) = \mathsf{spec}^{--}(d) + \frac{6}{7}(-\kappa_1) + \frac{5}{7}(-\kappa_2)$$
$$\mathsf{spec}^{--}(a) = \mathsf{spec}^{--}(d) + \frac{1}{7}(-\kappa_1) + \frac{2}{7}(-\kappa_2)$$
$$\mathsf{spec}^{--}(b) = \mathsf{spec}^{--}(d) + \frac{3}{7}(-\kappa_1) + \frac{6}{7}(-\kappa_2).$$

## A.5. *Pitch-related properties*

*Pitch*: The pitch value of any linear combination of generators (i.e. any element of $\mathbf{Z}[X]$) can be directly calculated as the linear combination of the pitch values of the generators specified above. This way, we can calculate the following values. (Figure A3 depicts these values on the pitch circle.)

$$\mathsf{pitch}\colon \mathsf{spec}(c) \longmapsto 0.830\ldots$$
$$\mathsf{spec}(d) \longmapsto 0$$
$$\mathsf{spec}(e) \longmapsto 0.152\ldots$$
$$\mathsf{spec}(f) \longmapsto 0.245\ldots$$
$$\mathsf{spec}(g) \longmapsto 0.415\ldots$$

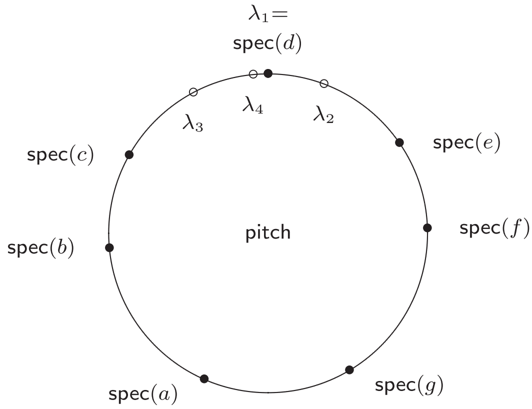



Figure A3. Tones and limits in the pitch domain.

$$\begin{aligned}\mathsf{spec}(a) &\longmapsto 0.567\ldots\\ \mathsf{spec}(b) &\longmapsto 0.737\ldots\\ \lambda_2 &\longmapsto 0.059\ldots\\ \lambda_3 &\longmapsto 0.923\ldots\\ \lambda_4 &\longmapsto 0.982\ldots\end{aligned}$$

*Between*: Loosely speaking, $\alpha_2$ is between $\alpha_1$ and $\alpha_3$ if it follows $\alpha_1$ and is followed by $\alpha_3$ on the pitch circle (clockwise). For example, $\lambda_4$ is between $\lambda_3$ and $\lambda_2$ but is not between $\lambda_2$ and $\lambda_3$.

*Closure condition*: Figure A3 clearly visualizes that the extremity is the only tone whose spec value is positioned within the cluster of the limits. Therefore, the GTS $S$ is tight. (And so are all its neighbouring GTSs.)

*Span*: The span of any pair of tones can be easily counted on the pitch circle (clockwise); e.g. $\mathsf{span}(c, e) = 2, \mathsf{span}(d, g) = 3, \mathsf{span}(e, c) = 5$, etc.

*Symmetry condition*: It can be directly verified that the first condition from Definition 2.21 holds. Therefore, the GTS $S$ is semi-WF. And due to the fact that the limits are positioned in a cluster the span is invariant in the neighbouring GTSs. It means that if we denote $\mathsf{span}_N$ the span function in any neighbouring $GTS$ then $\mathsf{span}(t, u) = \mathsf{span}_N(t, u)$ for any pair of tones $t, u \in T$. Therefore, the GTS is also WF.

## Appendix 3. Proof of the main theorem

For the entire section, we consider a two-dimensional demarcated GTS $S = (T, X, K, \mathsf{int}, \mathsf{spec}, \mathsf{pitch})$ with a set of generators $X = \{\xi_1, \xi_2\}$, a set of commas $K = \{\kappa_1, \kappa_2\}$, and extremity $e$. Before embarking on the main proof of the theorem, we first formulate a series of lemmas.

LEMMA A.1 *Let $S$ be tight. Consider a demarcated GTS $S' = (T', X, K, \mathsf{int}', \mathsf{spec}', \mathsf{pitch})$ having the same set of generators, the same set of commas and the same pitch function as $S$ and assume that $\mathsf{spec}(a) = \mathsf{spec}'(a')$ and $\mathsf{spec}(b) = \mathsf{spec}'(b')$ for $a, b \in T$ and $a', b' \in T'$. Then $(a, b)$ is a step in $S$ if and only if $(a', b')$ is a step in $S'$.*

*Proof* Due to the symmetry of the statement, it is sufficient to prove one implication. We proceed by contradiction. Therefore, we assume that $(a, b)$ is a step in $S$ and $(a', b')$ is not a step in $S'$. It means that there is a tone $c' \in T'$ between $a'$ and $b'$. Denote $\alpha = \mathsf{spec}(a) = \mathsf{spec}'(a')$, $\beta = \mathsf{spec}(b) = \mathsf{spec}'(b')$ and $\gamma' = \mathsf{spec}(c')$.

As $\kappa_1$ and $\kappa_2$ are linearly independent in $\mathbf{Z}[X]$, for any element $\mu \in \mathbf{Z}[X]$, we may consider the real numbers $r_1(\mu)$ and $r_2(\mu)$ for which:

$$\mu = \mathsf{spec}(e) + r_1(\mu)\kappa_1 + r_2(\mu)\kappa_2.$$

Let $e'$ be the extremity of $S'$ and let $\epsilon' = \mathsf{spec}'(e')$. As the intersection of $\mathsf{spec}[T]$ and $\mathsf{spec}'[T']$ is non-empty, we have that $-1 < r_1(\epsilon'), r_2(\epsilon') < 1$. We will assume that both $r_1(\epsilon')$ and $r_2(\epsilon')$ are non-positive, i.e.

$$-1 < r_i(\epsilon') \leq 0, \quad \text{for } i = 1, 2. \tag{A1}$$

All the other three combinations of signs can be handled analogically.

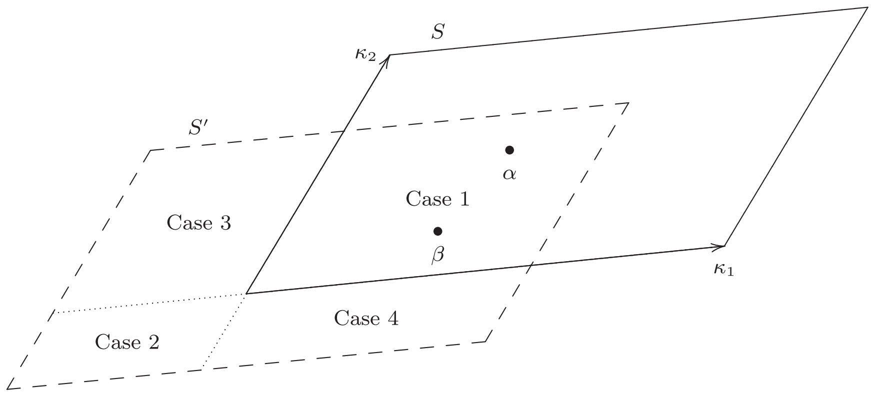



Figure A4. Four possible cases for the position of $\gamma'$.

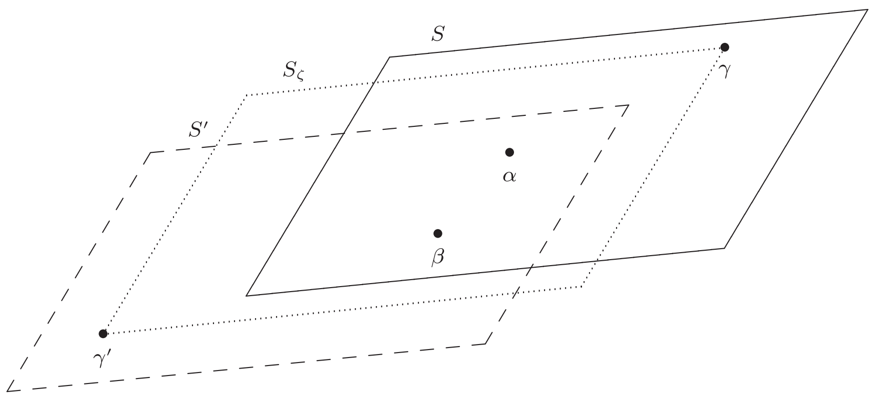



Figure A5. Contradiction in Case 2.



Now, take the real numbers $r_1(\gamma')$ and $r_2(\gamma')$. The demarcatedness of $S'$ implies $r_i(\epsilon') \leq r_i(\gamma') < r_i(\epsilon') + 1$. Combining these inequalities with (A1) yields $-1 < r_i(\gamma') < 1$ for $i = 1, 2$. Therefore, only the following four cases are possible (see also Figure A4).

*Case 1:* $0 \leq r_1(\gamma') < 1$ and $0 \leq r_2(\gamma') < 1$. In this case, the demarcatedness of $S$ yields existence of a tone $c \in T$ such that $\mathsf{spec}(c) = \gamma'$. Then, however, $c$ is between $a$ and $b$, which contradicts the fact that $(a, b)$ is a step in $S$. Therefore, this case is not possible.

*Case 2:* $-1 < r_1(\gamma') < 0$ and $-1 < r_2(\gamma') < 0$. In this case, $\gamma = \gamma' + \kappa + \lambda$ belongs to $\mathsf{spec}[T]$. Therefore, there is a tone $c \in T$ such that $\mathsf{spec}(c) = \gamma$. We consider also the transposition $S_\zeta$ of $S$, where $\zeta = \gamma - \mathsf{spec}(e)$ (Figure A5).

As $(a, b)$ is a step in $S$, we have $\lhd\,(\alpha, \beta, \gamma)$. On the other hand, $\gamma'$ is between $\alpha$ and $\beta$ (Figure A6). These two relations imply that $\beta$ is between $\gamma'$ and $\gamma$ and $\alpha$ is between $\gamma$ and $\gamma'$. And therefore, we have:

$$\lhd\,(\gamma, \alpha, \gamma') \tag{A2}$$

$$\lhd\,(\gamma', \beta, \gamma). \tag{A3}$$

In this case, relations (A2) and (A3) yield a contradiction with the CC for $S_\zeta$, which leads to a contradiction with the CC for its transposition, the GTS $S$.

*Case 3:* $-1 < r_1(\gamma) < 0$ and $0 \leq r_2(\gamma) < 1$. In this case, $\gamma = \gamma' + \kappa_1$ belongs to $\mathsf{spec}[T]$ and we have a tone $c \in T$ such that $\gamma = \mathsf{spec}(c)$.

Assume that both $\alpha$ and $\beta$ are positioned 'above' the line connecting $\gamma'$ and $\gamma$ (i.e. $r_2(\alpha) \geq r_2(\gamma)$ and $r_2(\beta) \geq r_2(\gamma)$). Then we consider the transposition $S_\zeta$ of $S$ similarly as in Case 2 and arrive at a contradiction with the CC. We would get a similar contradiction if both $\alpha$ and $\beta$ were positioned 'below' the line (i.e. if $r_2(\alpha) < r_2(\gamma)$ and $r_2(\beta) < r_2(\gamma)$).

Therefore, one of $\alpha$ and $\beta$ must be situated 'above' and the other 'below' the line. Without loss of generality, assume that $\alpha$ is 'above', $r_2(\alpha) \geq r_2(\gamma)$, and $\beta$ 'below', $r_2(\beta) < r_2(\gamma)$. This situation is shown in Figure A7.

Take the element

$$\delta = \gamma' + (\gamma - \beta).$$

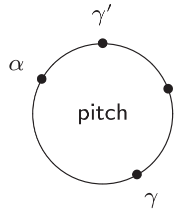



Figure A6. Contradiction in Case 2: the pitch domain.

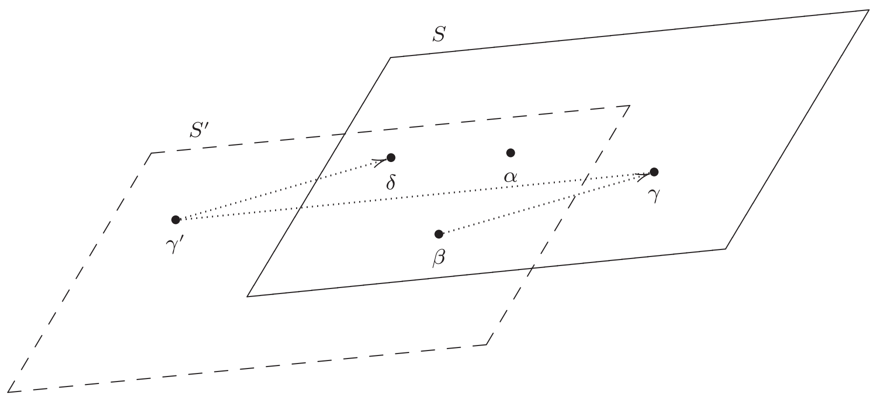



Figure A7. A 'dual' element $\delta$ to $\beta$ 'above' the comma $\gamma - \gamma'$.



Now both $\delta$ and $\alpha$ are 'above' the line connecting $\gamma'$ and $\kappa$:

$$r_2(\delta) = r_2(\gamma') + (r_2(\gamma) - r_2(\beta)) > r_2(\gamma') = r_2(\gamma).$$

Similarly as in Case 2, it can be proved that relations (A2) and (A3) hold also in this case. Thus, $\alpha$ is between $\gamma$ and $\gamma'$ and $\beta$ is between $\gamma'$ and $\gamma$. We will show that $\delta$ is between $\gamma'$ and $\gamma$, as well. As per definition, $\lhd(\gamma', \beta, \gamma)$ implies:

$$\mathsf{size}(\gamma', \gamma) = \mathsf{size}(\gamma', \beta) + \mathsf{size}(\beta, \gamma). \tag{A4}$$

From the definition of $\delta$, it follows that:

$$\mathsf{size}(\gamma', \beta) = \mathsf{size}(\delta, \gamma) \tag{A5}$$

$$\mathsf{size}(\beta, \gamma) = \mathsf{size}(\gamma', \delta) \tag{A6}$$

By combining (A4)–(A6), we obtain the following equation.

$$\mathsf{size}(\gamma', \gamma) = \mathsf{size}(\gamma', \delta) + \mathsf{size}(\delta, \gamma).$$

Thus, $\delta$ is between $\gamma'$ and $\gamma$. Therefore, applying also the fact that $\alpha$ is between $\gamma$ and $\gamma'$ and the fact that both $\alpha$ and $\delta$ are 'above' the line connecting $\gamma'$ and $\gamma$, we can arrive at a contradiction with the CC for $S_\zeta$ and also $S$ in a similar way as we did above for Case 2.

*Case 4:* $0 \le r_1(\gamma) < 1$ and $-1 < r_2(\gamma) < 0$. In this case, one may proceed analogically as in the previous one. This finishes the proof of lemma. ■

LEMMA A.2 *Let S be tight. Then any step in S is a step in all neighbouring GTSs.*

*Proof* First, consider that the neighbouring GTS switches just one comma, i.e. we have a mapping $N: K \to \{-1, 1\}$ such that $N(\kappa) = 1$ and $N(\lambda) = -1$ for $\{\kappa, \lambda\} = K$. Assume that $(a, b)$ is not a step in $S_N$, i.e. there is a tone $c \in T$ between $a$ and $b$ in $S_N$. We will show that this leads to a contradiction.

Denote $\alpha = \mathsf{spec}(a)$, $\beta = \mathsf{spec}(b)$, and $\gamma = \mathsf{spec}(c)$. As $N$ changes only the sign of the comma $\kappa$, the generation function of a tone $t$ is different in $S_N$ if and only if $t$ is a $\lambda$-edge tone. In that case, $\mathsf{spec}_N(t) = \mathsf{spec}(t) + \kappa$. There are eight theoretical combinations for $a$, $b$, and $c$ to be or not to be $\lambda$-edge tones. If all them were or were not $\lambda$-edge tones simultaneously, then $c$ would be between $a$ and $b$ in $S_N$, as well. This would yield a contradiction. Therefore, only the following six cases remain.

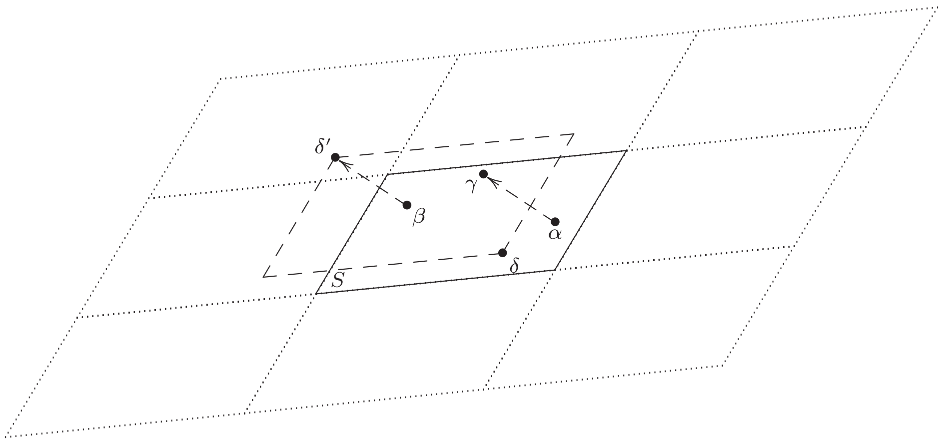



Figure A8. A possible position of $\delta'$.

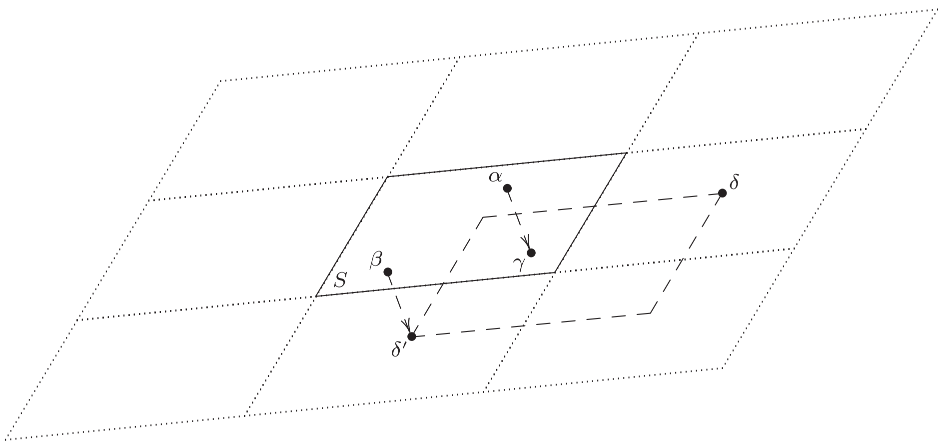



Figure A9. Another possibility for the position of $\delta'$.

In three of these cases, exactly one of the tones is a λ-edge tone. Assume that it is generated in $S$ by $\psi$. Then we may choose $\{\phi, \chi, \psi\} = \{\alpha, \beta, \gamma\}$ in such a way that $\lhd\, (\phi, \chi, \psi)$ and $\lhd\, (\phi, \psi + \kappa, \chi)$. This implies $\lhd\, (\psi, \phi, \psi + \kappa)$ and $\lhd\, (\psi + \kappa, \chi, \psi)$. Following the same ideas as in the proof of Lemma A.1, we may distinguish whether both $\phi$ and $\chi$ are 'on the same side' of the line connecting $\psi$ and $(\psi + \kappa)$. If yes, we directly construct a transposition of $S$ where the CC is not met. If not, we first take the 'dual' element $\omega = 2\psi + \kappa - \chi$ and then construct the transposition of $S$ contradicting the CC.

For the last three cases, exactly two of the tones are λ-edge tones. We proceed in the same way with the only difference that $\{\phi - \kappa, \chi - \kappa, \psi\} = \{\alpha, \beta, \gamma\}$, where $\psi$ is the value of the generation function of the non-λ-edge tone.

We proved that the lemma holds for neighbours which switch just one comma. The last possibility to consider is that both commas are changed in the neighbour, i.e. $N(\kappa_1) = -1$ and $N(\kappa_2) = -1$. In that case, we apply twice the statement we have just verified: If $(a, b)$ is a step in $S$ then it is a step in $S_M$ for $M\colon \kappa_1 \mapsto -1,\ \kappa_2 \mapsto 1$. This, in turn, implies that $(a, b)$ is also a step in $S_N$. ■

Lemma A.3 *Let $S$ be tight. Consider tones $a, b, c, d \in T$ such that $\mathsf{int}(a, c) = \mathsf{int}(b, d) = j$. Then $(a, b)$ is a step if and only if $(c, d)$ is a step.*

*Proof* Denote $\alpha = \mathsf{spec}(a)$, $\beta = \mathsf{spec}(b)$, $\gamma = \mathsf{spec}(c)$, and $\delta = \mathsf{spec}(d)$. Further, consider $\iota = \gamma - \alpha$ and $\delta' = \beta + \iota$. It is easy to see that $\delta'$ lies in one of the nine closest parallelograms with the parallelogram of $S$ at the centre. For each of these nine possibilities (two of them are shown in Figures A8 and A9), one may draw a parallelogram of same shape in such a way that $\delta$ and $\delta'$ are its corners and $\gamma$ lies in it.

Let $S_3$ denote the GTS delimited by this parallelogram and having the extremity generated as $\delta$, and similarly, let $S_2$ denote the neighbouring GTS with the extremity generated as $\delta'$. Finally, let $S_1$ denote the $\iota$-transposition of $S$ (Figure A10).

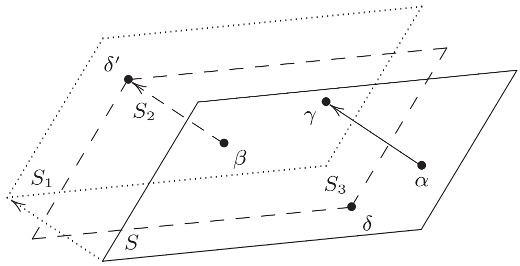



Figure A10. Visualization of the proof of Lemma A.3.

Since $(a, b)$ is a step in $S$ it is also a step in $S_1$ generated as $\gamma$ and $\delta'$. Applying Lemma A.1 to GTSs $S_1$ and $S_2$, we obtain that $\gamma$ and $\delta'$ make a step also in $S_2$. Then we apply Lemma A.2 to GTSs $S_2$ and $S_3$ and we get that $\gamma$ and $\delta$ generate a step in $S_3$. Finally, applying Lemma A.3 once more gives that $\gamma$ and $\delta$ make a step in $S$, as well. ■

Lemma A.4 *Consider a comma $\kappa$ and two tones $t, u$ that are inner tones or $\kappa$-edge tones. If $\lhd\ (\mathsf{spec}(e), \mathsf{spec}(t), \mathsf{spec}(e) + \kappa)$ and $\lhd\ (\mathsf{spec}(e) + \kappa, \mathsf{spec}(u), \mathsf{spec}(e))$, then $S$ is not WF.*

*Proof* The span of $(t, e)$ is not invariant for neighbouring GTSs, which yields a contradiction with the WF. ■

*Proof of Theorem 3.1* (Tight $\Rightarrow$ WF) First, we will show that the CC implies the well-formedness for demarcated two-dimensional GTSs (here the assumption of existence of at least one inner tone is not necessary). Consider any tones $t_1, u_1, t_2, u_2 \in T$ such that $\mathsf{int}(t_1, u_1) = \mathsf{int}(t_2, u_2)$.

Denote $n = \mathsf{span}(t_1, u_1)$. This means that we have $(n + 1)$ elements $t_1 = t_1^0, t_1^1, \ldots, t_1^n = u_1$ between $t_1$ and $u_1$ such that $(t_1^{i-1}, t_1^i)$ are steps for all $i = 1, \ldots, n$.

Now take the interval $j = \mathsf{int}(t_1, t_2)$ and consider the elements $t_2^i$ for $i = 0, \ldots, n$ such that $\mathsf{int}(t_1^i, t_2^i) = j$. It is easy to see that $t_2^0 = t_2$ and $t_2^n = u_2$.

Lemma A.3 implies that $(t_2^{i-1}, t_2^i)$ are steps for all $i = 1, \ldots, n$ and Lemma A.2 implies that they are steps in any neighbouring GTS. Therefore, for any mapping $N\colon K \to \{-1, 1\}$, we have that $\mathsf{span}_N(t_2, u_2) = n$. According to Definition 2.7, this means that $S$ is well-formed. This finished the proof of the first implication.

(WF $\Rightarrow$ Tight) We will prove the other implication by contradiction. Therefore, assume that $S$ is WF and loose. Then, for some $m, n \in T$ and $\kappa \in K$, we have:

$$\lhd\ (\mathsf{spec}(e), \mathsf{spec}(m), \mathsf{spec}(e) + \kappa) \quad \text{and} \quad \lhd\ (\mathsf{spec}(e) + \kappa, \mathsf{spec}(n), \mathsf{spec}(e)).$$

As $S$ contains at least one inner tone, we may assume that one of the tones $m$ and $n$ is inner. Without loss of generality, we may assume that $n$ is an inner tone. Applying Lemma A.4, we obtain that $m$ is a $\lambda$-edge tone where $\{\kappa, \lambda\} = K$.

Consider two sets: $M_\lambda = \{k \in T \mid \mathsf{spec}(k) - \mathsf{spec}(e) = r\lambda, r \in \mathbf{R}\}$ and $T_\lambda = \{k \in T \mid \mathsf{spec}(k) - \mathsf{spec}(n) = r\lambda, r \in \mathbf{R}\}$. $M_\lambda$ is a set of all $\lambda$-edge tones and $T_\lambda$ is the 'parallel' set containing the tone $n$. Denote $r_0 = \min\{r \in (0, 1) \mid \mathsf{spec}(e) + r\lambda \in M_\lambda\}$ and $m_0 = \mathsf{spec}^{-1}(\mathsf{spec}(e) + r\lambda)$. Then $M_\lambda$ and $T_\lambda$ are tones of one-dimensional GTSs $S_m$ and $S_n$, respectively, generated by $\eta = \mathsf{spec}(m_0) - \mathsf{spec}(e) \in \mathbf{Z}[X]$. The well-formedness of $S$ implies the well-formedness of these two one-dimensional GTSs.

From the results of Carey and Clampitt, we know that WF one-dimensional GTSs contain steps of up to two different sizes. It is easy to see that the difference between the larger and the smaller steps equals the comma. Denote $s_l$, $s_s$, and $c = \mathsf{size}(\mathsf{spec}(e), \mathsf{spec}(e) + \lambda)$ the sizes of the larger step, the smaller step, and the comma, respectively. The same sizes apply in $S_n$, $S_m$, and also in $S'_m$, the neighbouring GTS of $S_m$.

Without loss of generality, we may assume that $(m, e)$ or $(e, m)$ is a step in $S_m$. If $(m, e)$ is a step, then $\mathsf{size}(\mathsf{spec}(e), \mathsf{spec}(e) + \kappa)$ is larger then $s_l$ and there is at least one tone of $S_n$ (i.e. an inner tone) between $\mathsf{spec}(e)$ and $\mathsf{spec}(e) + \kappa$. Then Lemma A.4 leads to a contradiction. Therefore, $(e, m)$ is a step in $S_m$.

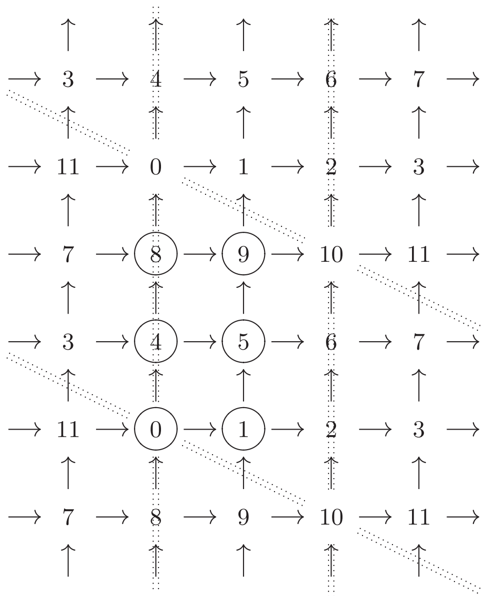
Taking into account again Lemma A.4, there are only two possibilities for the position of $\mathsf{spec}(e) + \lambda$.

*Case 1:* $\lhd\ (\mathsf{spec}(e), \mathsf{spec}(e) + \lambda, \mathsf{spec}(m))$. In this case, $(e, m)$ is a step also in the neighbouring $S'_m$ with lesser or equal size (of the corresponding intervals) than in $S_m$. Therefore $(e, m)$ is the larger step in $S_m$ and we have $\mathsf{size}(\mathsf{spec}(e), \mathsf{spec}(e) + \kappa) \geq s_l$. Then there is a tone of $S_n$ between $\mathsf{spec}(e)$ and $\mathsf{spec}(e) + \kappa$, and Lemma A.4 leads to a contradiction.

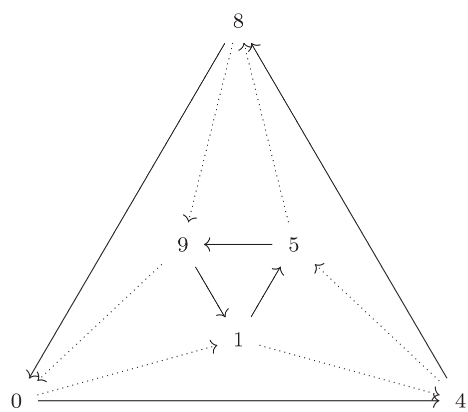
*Case 2:* $\lhd\ (\mathsf{spec}(e) + \lambda, \mathsf{spec}(e), \mathsf{spec}(m))$. Finally, in this case, $(e, m)$ is the smaller step in $S_m$ and it is the larger step in $S'_m$. Therefore, there is a tone of $S_n$ either between $\mathsf{spec}(e) + \lambda$ and $\mathsf{spec}(e)$ or between $\mathsf{spec}(e)$ and $\mathsf{spec}(m)$. In either case, Lemma A.4 gives a contradiction. This finishes the proof of the theorem. ■

## Appendix 4. Additional figures

This appendix contains additional figures illustrating two-dimensional WF GTSs and the related g-*Tonnetze* discussed in Section 4: the hexatonic scale, the octatonic scale, and the chromatic scale as generated by the semitone and the tritone.

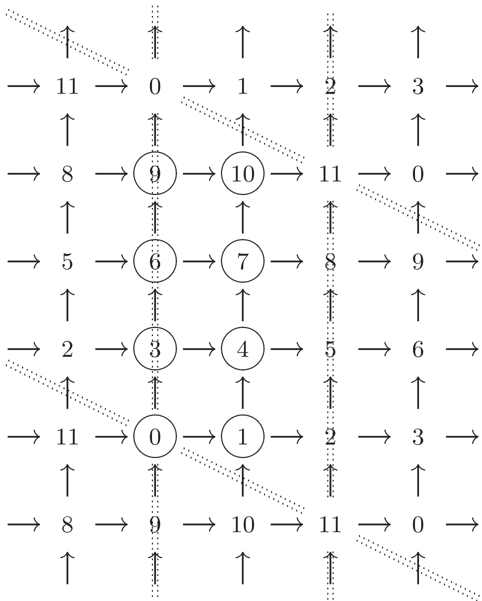



(a) GTS

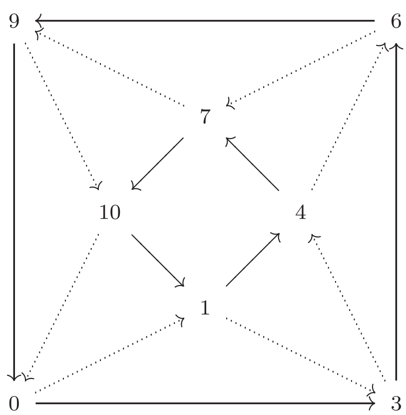



(b) g-*Tonnetz*

Figure A11. The hexatonic scale as a two-dimensional WF GTS.

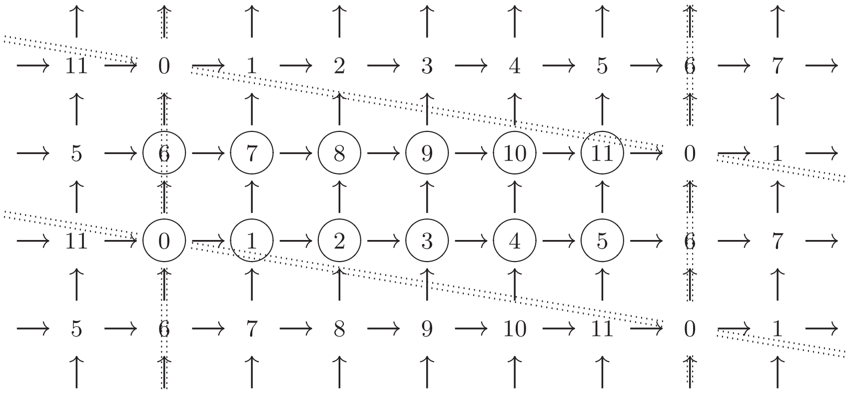



(a) GTS

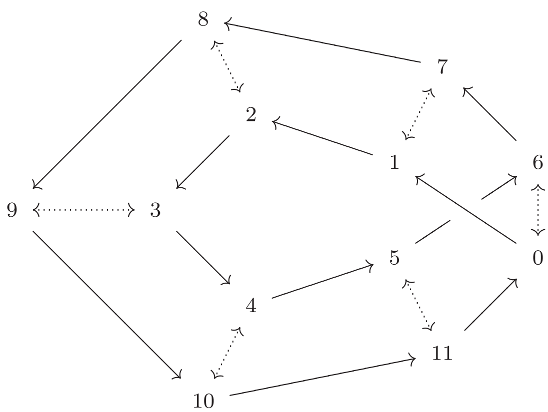



(b) g-*Tonnetz*

Figure A12. The octatonic scale as a two-dimensional WF GTS.




(a) GTS




(b) g-*Tonnetz*

Figure A13. The chromatic scale as a WF GTS generated by the semitone and the tritone.